Classes of Operator Forms

A.S.Baluck, S.F.Vinokurov
Irkutsk State University, Russia
sacha@hotmail.ru, vin@math.isu.ru

Abstract

The paper presents a classification of the classes of operator canonical forms of Boolean functions. These representations extend well-known exclusive-or sum-of-products expressions (ESOPs). We consider constructing methods and complexities of operator representations.

1 Introduction

This paper continues our research of operator forms of Boolean functions. Our aim is to construct a hierarchy of classes of these forms.

The main idea of operator forms is based on the following representation. Some bases of F_n can be represented as operator images of some functions. F_n denotes the space of all n-variable Boolean functions.

We introduce the class of operators and construct bundles of operators. In [3, 5], there are some generating criteria of a base of F_n by operator images of a function f. The choice of the function f and the bundle determines the canonical forms. In particular, if we put $f = x_1 \cdot \ldots \cdot x_n$, we obtain classes of ESOP.

In the paper we consider classes such as the following conditions hold: i) the classes have 'good' definitions; ii) there exist formulas to determine coefficients in representations; iii) bounds of Shannon function were found.

We compare our hierarchy with the well-known Green/Sasao hierarchy [1] and Inclusive Forms [2].

The size of this paper does not permit to present proofs and comparisons with some hierarchies [9].

2 Background

We use the following notation and abbreviations:
— vector of variables is denoted as $\tilde{x} = (x_1, ..., x_n)$;
— vector of constants is denoted as $\tilde{\tau} = (\tau_1, ..., \tau_n)$, where $\tau_i \in \{0, 1\}$, $\tilde{0} = (0, ..., 0)$, $\tilde{1} = (1, ..., 1)$,
— vector $(\bar{x}_1, ..., \bar{x}_n)$ is denoted by $\bar{\tilde{x}}$;
— symbol $\sum$ are used for summation modulo 2.

A sequence $\mathtt{t} = \mathtt{t}_1 ... \mathtt{t}_n$ with components $\mathtt{t}_i \in \{\mathtt{e}, \mathtt{p}, \mathtt{d}\}$ is called an *operator*; here n is called the *dimension* of operator $\mathtt{t}$ and denoted by $\dim \mathtt{t}$. An operator $\mathtt{t} = \mathtt{t}_1 ... \mathtt{t}_n$ regarded as a map $\mathtt{t} : F_n \to F_n$ is defined by the rule $\mathtt{t}g(\tilde{x}) = g_n(\tilde{x})$, where $g_0(\tilde{x}) = g(\tilde{x})$ and

$$g_i(\tilde{x}) = \begin{cases} g_{i-1}(\tilde{x}) & \text{if } \mathtt{t}_i = \mathtt{e} \\ g_{i-1}(x_1, ..., x_{i-1}, \bar{x}_i, x_{i+1}, ..., x_n) & \text{if } \mathtt{t}_i = \mathtt{p} \\ \partial g_{i-1}/\partial x_i & \text{if } \mathtt{t}_i = \mathtt{d}. \end{cases}$$

Note that $\partial g/\partial x_i$ is called the *derivative* of g with respect to a variable x_i and is defined as

$$\partial g/\partial x_i = g(\tilde{x}) \oplus g(x_1, ..., x_{i-1}, \bar{x}_i, x_{i+1}, ..., x_n).$$

Example Consider the operator $\mathtt{t} = \mathtt{epd}$ and the function $g(x_1, x_2, x_3) = x_1 \vee x_2 \vee x_3$. We have

$$g_0 = x_1 \vee x_2 \vee x_3, \quad g_1 = x_1 \vee x_2 \vee x_3, \quad g_3 = x_1 \vee \bar{x}_2 \vee x_3, \quad g_4 = (x_1 \vee \bar{x}_2 \vee x_3) \oplus (x_1 \vee \bar{x}_2 \vee \bar{x}_3) = \bar{x}_1 x_2.$$

Thus we have $\mathtt{epd}(g) = \bar{x}_1 x_2$.

A sequence $\mathsf{T} = (\mathrm{t}^{\tilde{0}}, ..., \mathrm{t}^{\tilde{\tau}}, ..., \mathrm{t}^{\tilde{1}})$ consisting of 2^n operators with the same dimensions is called a *bundle of operators*; here n is called the *dimension* of the bundle and is denoted by $\dim \mathsf{T}$.

A bundle of operators $(\mathrm{t}^{\tilde{0}}, ..., \mathrm{t}^{\tilde{\tau}}, ..., \mathrm{t}^{\tilde{1}})$ is called a *base bundle* if there exists a function $g(x_1, ..., x_n)$ such that $\{\mathrm{t}^{\tilde{0}}g, \ldots, \mathrm{t}^{\tilde{\tau}}g, \ldots, \mathrm{t}^{\tilde{1}}g\}$ is a basis for F_n, i.e., for any function $f \in F_n$ there exists a unique representation

$$f = \sum_{\tilde{\tau}} \alpha_{\tilde{\tau}} \mathrm{t}^{\tilde{\tau}} g, \text{ where } \alpha_{\tilde{\tau}} \in \{0, 1\}. \qquad (*)$$

This representation is called a *canonical operator form*.

By definition, put

$$L^g_{\mathsf{T}}(f) = \sum_{\tilde{\tau}} \alpha_{\tilde{\tau}}.$$

Let K be a class of base bundles; then define the *complexity* $L^g_K(f)$ by the rule

$$L^g_K(f) = \min_{\mathsf{T} \in K} L^g_{\mathsf{T}}(f)$$

and define *Shannon function* by the rule

$$L^g_K(n) = \max_{f \in F_n} L^g_K(f).$$

Example Suppose $g = x_1x_2x_3$ is a function, $\mathsf{T} = (\mathrm{eee}, \mathrm{eep}, \mathrm{epe}, \mathrm{epp}, \mathrm{pee}, \mathrm{pep}, \mathrm{ppe}, \mathrm{ppp})$ is a bundle of operators. Then we have

$$\begin{array}{cccc} \mathrm{eee}(g) = x_1x_2x_3 & \mathrm{eep}(g) = x_1x_2\bar{x}_3 & \mathrm{epe}(g) = x_1\bar{x}_2x_3 & \mathrm{epp}(g) = x_1\bar{x}_2\bar{x}_3 \\ \mathrm{pee}(g) = \bar{x}_1x_2x_3 & \mathrm{pep}(g) = \bar{x}_1x_2\bar{x}_3 & \mathrm{ppe}(g) = \bar{x}_1\bar{x}_2x_3 & \mathrm{ppp}(g) = \bar{x}_1\bar{x}_2\bar{x}_3. \end{array}$$

Theorem 2.1 [3] Suppose $\mathsf{T} = (\mathrm{t}^{\tilde{0}}, ..., \mathrm{t}^{\tilde{\tau}}, ..., \mathrm{t}^{\tilde{1}})$ is a base bundle; then $\mathrm{t}^{\tilde{0}}g, \ldots, \mathrm{t}^{\tilde{\tau}}g, \ldots, \mathrm{t}^{\tilde{1}}g$ is a basis for F_n iff $\sum_{\tilde{\tau}} g(\tilde{\tau}) = 1$.

A function is called a *base function* if $\sum_{\tilde{\tau}} g(\tilde{\tau}) = 1$.

Theorem 2.2 [3] Suppose K is a class of base bundles, g and h are base functions; then

$$L^g_K(n) = L^h_K(n).$$

By definition, put $L_K(n) = L^g_K(n)$, where g is any base function. If $g = x_1 \cdot \ldots \cdot x_n$ then we write $L^{\&}_K(f)$ instead of $L^g_K(f)$.

A bundle $\mathsf{T} = (\mathrm{t}^{\tilde{0}}, ..., \mathrm{t}^{\tilde{1}})$ is called *two-generated* if there exist operators a and b such that $\mathrm{a}_i \neq \mathrm{b}_i$ and

$$\mathrm{t}^{\tilde{\tau}}_i = \begin{cases} \mathrm{a}_i & \text{if } \tau_i = 0 \\ \mathrm{b}_i & \text{if } \tau_i = 1. \end{cases}$$

This bundle is denoted by $D(\mathrm{a}, \mathrm{b})$.

For example, the bundle $(\mathrm{dpe}, \mathrm{dpd}, \mathrm{dde}, \mathrm{ddd}, \mathrm{ppe}, \mathrm{ppd}, \mathrm{pde}, \mathrm{pdd})$ is two-generated by dpe and pdd.

A bundle $\mathsf{T} = (\mathrm{t}^{\tilde{0}}, ..., \mathrm{t}^{\tilde{1}})$ is called *one-generated* by an operator $\mathrm{a} = \mathrm{a}_1 ... \mathrm{a}_n$ if

$$\mathrm{t}^{\tilde{\tau}}_i = \mathrm{a}_i \quad \text{if } \tau_i = 0, \qquad \mathrm{t}^{\tilde{\tau}}_i \neq \mathrm{a}_i \quad \text{if } \tau_i = 1.$$

As an example, the bundle $(\mathrm{dpe}, \mathrm{dpd}, \mathrm{dde}, \mathrm{ded}, \mathrm{ppe}, \mathrm{ppd}, \mathrm{eee}, \mathrm{pdp})$ is one-generated by dpe.

Let $\mathsf{T} = (\mathrm{t}^{\tilde{0}}, ..., \mathrm{t}^{\tilde{1}})$ be a two-generated bundle. It follows that there exist two operators b and c such that $\mathsf{T} = D(\mathrm{b}, \mathrm{c})$. Let a be an operator such that $\mathrm{a}_i \neq \mathrm{b}_i$ and $\mathrm{a}_i \neq \mathrm{c}_i$ for any i. Let $\mathsf{B} = (\mathrm{b}^{\tilde{0}}, ..., \mathrm{b}^{\tilde{1}})$ be a bundle and let $\tilde{\sigma}$ be a vector such that $\mathrm{b}^{\tilde{\tau}} = \mathrm{t}^{\tilde{\tau}}$ for all $\tilde{\tau} \neq \tilde{\sigma}$ and $\mathrm{b}^{\tilde{\sigma}} = \mathrm{a}$. Then the bundle B is called an *extension of the bundle* T *by the operator* a.

In other words, we put the operator a in the bundle T instead of one of the operators $\mathrm{t}^{\tilde{\tau}}$.

For instance, the bundle $(\mathrm{dpe}, \mathrm{dpd}, \mathrm{dde}, \mathrm{ddd}, \mathrm{ppe}, \mathrm{eep}, \mathrm{pde}, \mathrm{pdd})$ is an extension of the bundle $D(\mathrm{dpe}, \mathrm{pdd})$ by the operator eep.

Let $\mathsf{A} = (\mathfrak{a}^{\tilde{0}}, ..., \mathfrak{a}^{\tilde{1}})$ be a bundle such that $\dim \mathsf{A} = n$, let

$$\{\mathsf{B}_{\tilde{\tau}} \mid \mathsf{B}_{\tilde{\tau}} = (\mathfrak{b}_{\tilde{\tau}}^{\tilde{0}}, ..., \mathfrak{b}_{\tilde{\tau}}^{\tilde{1}}), \quad \tilde{\tau} = (\tau_1, ..., \tau_n), \quad \tau_i \in \{0,1\}, \quad \dim \mathsf{B}_{\tilde{\tau}} = m\}$$

be a set consisting of 2^n bundles; then the bundle $\mathsf{C} = (\mathfrak{c}^{\tilde{0}}, ..., \mathfrak{c}^{\tilde{\sigma}}, ..., \mathfrak{c}^{\tilde{1}})$ is called the *wreath product* of the bundles $\mathsf{B}_{\tilde{0}}, ..., \mathsf{B}_{\tilde{1}}$ by the bundle A if $\dim \mathsf{C} = n + m$ and

$$\mathfrak{c}_1^{\tilde{\sigma}} \dots \mathfrak{c}_n^{\tilde{\sigma}} = \mathfrak{a}^{\tilde{\tau}}, \quad \mathfrak{c}_{n+1}^{\tilde{\sigma}} \dots \mathfrak{c}_{n+m}^{\tilde{\sigma}} = \mathfrak{b}_{\tilde{\tau}}^{\tilde{\upsilon}}, \quad \text{where } \tilde{\tau} = (\sigma_1, ..., \sigma_n) \text{ and } \tilde{\upsilon} = (\sigma_{n+1}, ..., \sigma_{n+m}).$$

The wreath product of bundles $\mathsf{B}_{\tilde{0}}, ..., \mathsf{B}_{\tilde{1}}$ by the bundle A is denoted by $W(\mathsf{A} \mid \mathsf{B}_{\tilde{0}}, ..., \mathsf{B}_{\tilde{1}})$.

Let $\mathsf{A} = (\mathfrak{a}^{\tilde{0}}, ..., \mathfrak{a}^{\tilde{\tau}}, ..., \mathfrak{a}^{\tilde{1}})$, $\mathsf{B} = (\mathfrak{b}^{\tilde{0}}, ..., \mathfrak{b}^{\tilde{\tau}}, ..., \mathfrak{b}^{\tilde{1}})$ be bundles. If there exists a permutation $I = (i_1, ..., i_n)$ such that $\mathfrak{b}_j^{\tilde{\tau}} = \mathfrak{a}_{i_j}^{\tilde{\tau}}$, then we say that the bundle B is a *permutation* of the bundle A and write $\mathsf{B} = I(\mathsf{A} \mid i_1, ..., i_n)$.

Example Take the bundles $\mathsf{A} = (\mathfrak{ee}, \mathfrak{ep}, \mathfrak{pe}, \mathfrak{dd})$, $\mathsf{B}_{(00)} = (\mathfrak{e}, \mathfrak{d})$, $\mathsf{B}_{(01)} = (\mathfrak{e}, \mathfrak{p})$, $\mathsf{B}_{(10)} = (\mathfrak{p}, \mathfrak{d})$, $\mathsf{B}_{(11)} = (\mathfrak{d}, \mathfrak{p})$. Then

$$C = W(\mathsf{A} \mid \mathsf{B}_{00}, \mathsf{B}_{01}, \mathsf{B}_{10}, \mathsf{B}_{11}) = (\mathfrak{eee}, \mathfrak{eed}, \mathfrak{epe}, \mathfrak{epp}, \mathfrak{pep}, \mathfrak{ped}, \mathfrak{ddd}, \mathfrak{ddp}),$$
$$I(\mathsf{C} \mid 3, 2, 1) = (\mathfrak{eee}, \mathfrak{dee}, \mathfrak{epe}, \mathfrak{ppe}, \mathfrak{pep}, \mathfrak{dep}, \mathfrak{ddd}, \mathfrak{pdd}).$$

Theorem 2.3 [3]
(1) The wreath product of base bundles by a base bundle is a base bundle.
(2) Permutation of a base bundle is a base bundle.

Using Theorem 2.3, we can recursively construct base bundles.

Define the set of functions $M_n = \{p_n, q_n, r_n\}$ by induction on n:

$$p_0 = 0, \quad q_0 = r_0 = 1,$$
$$p_n = x_n q_{n-1} \oplus \bar{x}_n r_{n-1}, \quad q_n = x_n r_{n-1} \oplus \bar{x}_n p_{n-1}, \quad r_n = x_n p_{n-1} \oplus \bar{x}_n q_{n-1}.$$

In particular,

$$p_1 = (11), \quad p_2 = (1001), \quad p_3 = (0111\,1110), \quad p_4 = (1110\,1001\,1001\,0111);$$
$$q_1 = (01), \quad q_2 = (1110), \quad q_3 = (1001\,0111), \quad q_4 = (0111\,1110\,1110\,1001);$$
$$r_1 = (10), \quad r_2 = (0111), \quad r_3 = (1110\,1001), \quad r_4 = (1001\,0111\,0111\,1110).$$

Further, define the set of functions M_n^* by the rule

$$M_n^* = \{f \mid f(x_1^{\sigma_1}, ..., x_n^{\sigma_n}) \in M_n, \text{ where } \sigma_i \in \{0,1\}\}.$$

Let us remark that the functions $\bar{p}_n$, $\bar{q}_n$, and $\bar{r}_n$ were used in [4].

3 Classes of Operator Bundles

3.1 $\mathfrak{a}$-Kronecker Classes

Let $\mathfrak{a} = \mathfrak{a}_1 ... \mathfrak{a}_n$ be an operator. By definition, put

$$K(\mathfrak{a}) = \{\mathsf{T} \mid \mathsf{T} = D(\mathfrak{a}, \mathfrak{b}), \text{ where } \mathfrak{b} = \mathfrak{b}_1 ... \mathfrak{b}_n \text{ is any operator such that } \mathfrak{b}_i \neq \mathfrak{a}_i\}.$$

The class $K(\mathfrak{a})$ is called $\mathfrak{a}$-*Kronecker class.*

Theorem 3.1 The following conditions hold:
(1) if $\mathsf{T} \in K(\mathfrak{a})$, then T is a base bundle;
(2) if $\mathsf{T} \in K(\mathfrak{a})$ and $f \in F_n$, then the coefficients of representation $(*)$ are given by

$$\alpha_{\tilde{\tau}} = \sum_{\tilde{\sigma}} h(\tilde{\sigma}) \cdot f(\tilde{\sigma}), \quad \text{where } h(\tilde{x}) = \mathfrak{t}^{\bar{\tilde{\tau}}} g(\bar{\tilde{x}});$$

(3) $L_{K(\mathfrak{a})}(n) = \lfloor \frac{2}{3} 2^n \rfloor$;
(4a) if $f \in F_n$ and n is odd, then $L^{\&}_{K(\mathfrak{d}...\mathfrak{d})}(f) = \lfloor \frac{2}{3} 2^n \rfloor$ iff $f \in M_n^*$;

(4b) if $f \in F_n$ and n is even, then $L^{\&}_{K(\mathtt{d}\ldots\mathtt{d})}(f) = \lfloor \frac{2}{3} 2^n \rfloor$ iff $f \in M^*_n \cup N$,
where $N = \{h \mid h = \bar{p}_n(x_1^{\sigma_1}, ..., x_n^{\sigma_n}),\ \text{where } \sigma_i \in \{0,1\}\}$.

Example Consider the operator $\mathtt{a} = \mathtt{ddd}$. By definition, $K(\mathtt{ddd})$ consists of the bundles:

$$\begin{array}{ll}
(\mathtt{ddd}, \mathtt{dde}, \mathtt{ded}, \mathtt{dee}, \mathtt{edd}, \mathtt{ede}, \mathtt{eed}, \mathtt{eee}), & (\mathtt{ddd}, \mathtt{ddp}, \mathtt{ded}, \mathtt{dep}, \mathtt{edd}, \mathtt{edp}, \mathtt{eed}, \mathtt{eep}), \\
(\mathtt{ddd}, \mathtt{dde}, \mathtt{dpd}, \mathtt{dpe}, \mathtt{edd}, \mathtt{ede}, \mathtt{epd}, \mathtt{epe}), & (\mathtt{ddd}, \mathtt{ddp}, \mathtt{dpd}, \mathtt{dpp}, \mathtt{edd}, \mathtt{edp}, \mathtt{epd}, \mathtt{epp}), \\
(\mathtt{ddd}, \mathtt{dde}, \mathtt{ded}, \mathtt{dee}, \mathtt{pdd}, \mathtt{pde}, \mathtt{ped}, \mathtt{pee}), & (\mathtt{ddd}, \mathtt{ddp}, \mathtt{ded}, \mathtt{dep}, \mathtt{pdd}, \mathtt{pdp}, \mathtt{ped}, \mathtt{pep}), \\
(\mathtt{ddd}, \mathtt{dde}, \mathtt{dpd}, \mathtt{dpe}, \mathtt{pdd}, \mathtt{pde}, \mathtt{ppd}, \mathtt{ppe}), & (\mathtt{ddd}, \mathtt{ddp}, \mathtt{dpd}, \mathtt{dpp}, \mathtt{pdd}, \mathtt{pdp}, \mathtt{ppd}, \mathtt{ppp}).
\end{array}$$

Suppose $g = x_1 \cdot x_2 \cdot x_3$, $f = (0110\,1101)$. Then f is represented in the following $\mathtt{ddd}$-Kronecker forms:

$$\begin{array}{ll}
f = x_3 \oplus x_2 \oplus x_1 \oplus x_1x_3 \oplus x_1x_2x_3, & f = 1 \oplus \bar{x}_3 \oplus x_2 \oplus x_1\bar{x}_3 \oplus x_1x_2 \oplus x_1x_2\bar{x}_3, \\
f = 1 \oplus x_3 \oplus \bar{x}_2 \oplus x_1 \oplus x_1\bar{x}_2x_3, & f = \bar{x}_3 \oplus \bar{x}_2 \oplus x_1 \oplus x_1\bar{x}_2 \oplus x_1\bar{x}_2\bar{x}_3, \\
f = 1 \oplus x_2 \oplus x_2x_3 \oplus \bar{x}_1 \oplus \bar{x}_1x_3 \oplus \bar{x}_1x_2x_3, & f = 1 \oplus x_2\bar{x}_3 \oplus \bar{x}_1\bar{x}_3 \oplus \bar{x}_1x_2 \oplus \bar{x}_1x_2\bar{x}_3, \\
f = x_3 \oplus \bar{x}_2 \oplus \bar{x}_2x_3 \oplus \bar{x}_1 \oplus \bar{x}_1\bar{x}_2x_3, & f = 1 \oplus \bar{x}_3 \oplus \bar{x}_2\bar{x}_3 \oplus \bar{x}_1 \oplus \bar{x}_1\bar{x}_2 \oplus \bar{x}_1\bar{x}_2\bar{x}_3.
\end{array}$$

We obtain $L^{\&}_{K(\mathtt{ddd})}(f) = 5 = \lfloor \frac{2}{3} 2^3 \rfloor$. On the other hand, we have $f(x_1, x_2, x_3) = q_3(x_1, \bar{x}_2, x_3)$. It follows that $f \in M^*_n$. Combining this with Theorem 3.3 we obtain $L^{\&}_{K(\mathtt{ddd})}(f) = 5$ again.

This example shows that $\mathtt{ddd}$-Kronecker class contains all well-known Reed-Muller forms with fixed polarity of 3-variable functions. It is obvious that $\mathtt{d}\ldots\mathtt{d}$-Kronecker classes contain all Fixed Polarity Reed-Muller forms [1].

3.2 Kronecker Class

The class of all two-generated bundles is called *Kronecker class* and is denoted by K.

Theorem 3.2 The following conditions hold:
(1) if $\mathsf{T} \in K$, then T is a base bundle;
(2) if $\mathsf{T} \in K$ and $f \in F_n$, then the coefficients of representation $(*)$ are given by

$$\alpha_{\tilde{\tau}} = \sum_{\tilde{\sigma}} h(\tilde{\sigma}) \cdot f(\tilde{\sigma}), \quad \text{where } h(\tilde{x}) = \mathtt{t}^{\overline{\tilde{\tau}}} g(\overline{\tilde{x}});$$

(3) $L_K(n) = \lfloor \frac{2}{3} 2^n \rfloor$;
(4) if $f \in F_n$, then $L^{\&}_K(f) = \lfloor \frac{2}{3} 2^n \rfloor$ iff $f \in M^*_n$.

Example Consider the function $\bar{p}_4 = (0001\,0110\,0110\,1000)$ and the bundle $\mathsf{T} = D(\mathtt{pppp}, \mathtt{eeee})$. It is obvious that $\mathtt{a} \in K$. We have

$$\bar{p}_4(x_1, x_2, x_3, x_4) = \bar{x}_1\bar{x}_2x_3x_4 \oplus \bar{x}_1x_2\bar{x}_3x_4 \oplus \bar{x}_1x_2x_3\bar{x}_4 \oplus x_1\bar{x}_2\bar{x}_3x_4 \oplus x_1\bar{x}_2x_3\bar{x}_4 \oplus x_1x_2\bar{x}_3\bar{x}_4.$$

¿From Theorem 3.1(4b) and Theorem 3.2(4) it follows that $L^{\&}_{K(\mathtt{dddd})}(\bar{p}_4) = \lfloor \frac{2}{3} 2^4 \rfloor = 10$ but $L^{\&}_K(\bar{p}_4) < 10$. This example shows that $L^{\&}_K(\bar{p}_4) \leq 6 < 10$.

It is easily shown that Kronecker class contains all well-known Kronecker Expansions [1]. Theorem 3.2(2) give a formula to compute their coefficients.

3.3 Free Kronecker class

Let FK be the class of bundles constructing by the rules

(i) $(\mathtt{e}, \mathtt{p}) \in FK$, $(\mathtt{p}, \mathtt{d}) \in FK$, $(\mathtt{d}, \mathtt{e}) \in FK$;

(ii) if $\mathsf{A} \in FK$ and $\mathsf{B}_{\tilde{0}}, ..., \mathsf{B}_{\tilde{1}} \in FK$, then wreath product of $\mathsf{B}_{\tilde{0}}, ..., \mathsf{B}_{\tilde{1}}$ by A belongs to FK;

(iii) if $\mathsf{A} \in FK$, then any permutation of A belongs to FK.

The class FK is called *Free Kronecker class.*

Theorem 3.3 The following conditions hold:
(1) if $\mathsf{T} \in FK$, then T is a base bundle;
(2) if $\mathsf{T} \in FK$ and $f \in F_n$, then the coefficients of representation $(*)$ are given by

$$\alpha_{\tilde{\tau}} = \sum_{\tilde{\sigma} \in N_{\tilde{\tau}}} f(\tilde{\sigma}),$$

where $N_{\tilde{\tau}} = \{\tilde{\sigma} \mid \sigma_i = 0 \text{ if } \mathrm{t}^{\tilde{\tau}_i} = \mathrm{e}, \quad \sigma_i = 1 \text{ if } \mathrm{t}^{\tilde{\tau}_i} = \mathrm{p}\}$ and $\tilde{\tau}_i = (tau_1, ..., \tau_{i-1}, \bar{\tau}_i, \tau_{i+1}, ..., \tau_n)$.
(3) $L_{FK}(n) = \frac{1}{2}2^n$;
(4) if $f \in F_n$, then $L^{\&}_{FK}(f) = \frac{1}{2}2^n$ iff $f \in M^*_n$.

Example Take the bundle $\mathsf{T} = (\mathrm{epp}, \mathrm{epd}, \mathrm{ppd}, \mathrm{ppe}, \mathrm{pdd}, \mathrm{ddd}, \mathrm{dde}, \mathrm{ede})$. It is clear that T is not two-generated. Let us show that $\mathsf{T} \in FK$.

Consider the bundles

$$\begin{array}{ll} \mathsf{B}_1 = W((\mathrm{d},\mathrm{e}) \mid (\mathrm{p},\mathrm{d}),(\mathrm{d},\mathrm{e})), & \mathsf{B}_2 = W((\mathrm{e},\mathrm{p}) \mid (\mathrm{p},\mathrm{d}),(\mathrm{d},\mathrm{e})), \\ \mathsf{B}_3 = I(\mathsf{B}_1 \mid 2,1), & \mathsf{B}_4 = W((\mathrm{p},\mathrm{d}) \mid \mathsf{B}_2,\mathsf{B}_3). \end{array}$$

Thus $\mathsf{T} = I(\mathsf{B}_4 \mid 2,1,3)$.

Suppose $g = x_1x_2x_3$, $f = (0110\,1101)$. Using T and g, we have the representation

$$f(x_1,x_2,x_3) = x_1\bar{x}_2\bar{x}_3 \oplus \bar{x}_1\bar{x}_2 \oplus \bar{x}_1 \oplus x_3.$$

Thus we have $L^{\&}_{FK}(f) \le 4 < L^{\&}_{K}(f) = 5$. Using Theorem 3.3(4), it follows that $L^{\&}_{FK}(f) = 4$.

It can be proved that Free Kronecker class contains all Free Kronecker Expansions [1].

3.4 $\mathfrak{a}$-Extended Classes

Let $\mathfrak{a}$ be an operator. The class of all extensions of two-generated bundles by the operator $\mathfrak{a}$ is called $\mathfrak{a}$-*Extended class* and is denoted by $E(\mathfrak{a})$.

Let us remark that not for any two-generated bundle there exist an extension of this bundle by the operator $\mathfrak{a}$.

Theorem 3.4 The following conditions hold:
(1) if $\mathsf{T} \in E(\mathfrak{a})$, then T is a base bundle;
(2) if $\mathsf{T} \in E(\mathfrak{a})$ and $f \in F_n$, then the coefficients of representation $(*)$ are given by

$$\alpha_{\tilde{\tau}} = \begin{cases} \sum\limits_{\tilde{\sigma}} h(\tilde{\sigma}) \cdot f(\tilde{\sigma}), & \text{if } \mathrm{t}^{\tilde{\tau}} \neq \mathfrak{a}, \text{ where } h(\tilde{x}) = \mathrm{t}^{\bar{\tilde{\tau}}} g(\bar{\tilde{x}}) \oplus \mathfrak{a}g(\bar{\tilde{x}}) \\ \sum\limits_{\tilde{\sigma}} h(\tilde{\sigma}) \cdot f(\tilde{\sigma}), & \text{if } \mathrm{t}^{\tilde{\tau}} = \mathfrak{a}, \text{ where } h(\tilde{x}) = \mathrm{t}^{\bar{\tilde{\tau}}} g(\bar{\tilde{x}}); \end{cases}$$

(3) $L_{E(\mathfrak{a})}(n) = \frac{1}{2}2^n$;
(4) if $f \in F_n$, then $L^{\&}_{E(\mathrm{d}...\mathrm{d})}(f) = \frac{1}{2}2^n$ iff $\frac{1}{2}2^n \le \sum\limits_{\tilde{\tau}} f(\tilde{\tau}) \le \frac{1}{2}2^n + 1$.

Example Let $\mathsf{T} = D(\mathrm{ddd}, \mathrm{ppp})$, let $\mathfrak{a} = \mathrm{eee}$. Consider the bundle

$$\mathsf{A} = (\mathrm{ddd}, \mathrm{ddp}, \mathrm{dpd}, \mathrm{dpp}, \mathrm{eee}, \mathrm{pdp}, \mathrm{ppd}, \mathrm{ppp}).$$

Clearly, the bundle A is an extension of the bundle T by the operator $\mathfrak{a}$. It follows that the bundle A belongs to eee-Extended class. Let $f = (0110\,1101)$ and $g = x_1x_2x_3$ be the functions. Using T and g, we get the following representation:

$$f = 1 \oplus \bar{x}_3 \oplus \bar{x}_2\bar{x}_3 \oplus \bar{x}_1 \oplus \bar{x}_1\bar{x}_2 \oplus \bar{x}_1\bar{x}_2\bar{x}_3.$$

If we use the bundle A, we have

$$f = \bar{x}_2 \oplus x_1x_2x_3 \oplus \bar{x}_1\bar{x}_3.$$

Thus we obtain $L^{\&}_{E(\mathrm{eee})}(f) \le 3$. Note that, using Theorem 3.4(4), we have $L_{E(\mathrm{ddd})}(f) = 4$.

3.5 Extended Class

The class of all extensions of two-generated bundles by any operator is called *Extended class* and is denoted by E.

It is obvious that $E = \bigcup\limits_{\mathfrak{a}} E(\mathfrak{a})$.

Theorem 3.5 The following conditions hold:
(1) if $\mathsf{T} \in E$, then T is a base bundle;

(2) if T is an extension by an operator $\mathfrak{a}$ and $f \in F_n$, then the coefficients of representation $(*)$ are given by

$$\alpha_{\tilde{\tau}} = \begin{cases} \sum_{\tilde{\sigma}} h(\tilde{\sigma}) \cdot f(\tilde{\sigma}), & \text{if } \mathfrak{t}^{\tilde{\tau}} \neq \mathfrak{a}, \text{ where } h(\tilde{x}) = \mathfrak{t}^{\bar{\tilde{\tau}}} g(\bar{\tilde{x}}) \oplus \mathfrak{a} g(\bar{\tilde{x}}) \\ \sum_{\tilde{\sigma}} h(\tilde{\sigma}) \cdot f(\tilde{\sigma}), & \text{if } \mathfrak{t}^{\tilde{\tau}} = \mathfrak{a}, \text{ where } h(\tilde{x}) = \mathfrak{t}^{\bar{\tilde{\tau}}} g(\bar{\tilde{x}}); \end{cases}$$

(3) $\lceil \frac{1}{3} 2^n \rceil \leq L_E(n) < \frac{1}{2} 2^n$;
(4) if $f \in M_n^*$, then $L_E^{\&}(f) = \lceil \frac{1}{3} 2^n \rceil$.

Let us remark that the functions $f \in M_n^*$ are not the most complex functions in Extended class if $n \geq 5$.

Example Consider the function $f = (0111\,0001\,0001\,1000\,1000\,0110\,0110\,1001)$. Let

$$\begin{array}{rl} \mathsf{T} = (& \mathsf{eeeee, ddddp, dddpd, dddpp, ddpdd, ddpdp, ddppd, ddppp} \\ & \mathsf{dpddd, dpddp, dpdpd, dpdpp, dppdd, dppdp, dpppd, dpppp} \\ & \mathsf{pdddd, pdddp, pddpd, pddpp, pdpdd, pdpdp, pdppd, pdppp} \\ & \mathsf{ppddd, ppddp, ppdpd, ppdpp, pppdd, pppdp, ppppd, ppppp} \quad) \end{array}$$

be an extension of the bundle $D(\mathsf{ddddd}, \mathsf{ppppp})$ by the operator eeeee. Using $g = x_1 x_2 x_3 x_4 x_5$ and T, we have the representation

$$f = \bar{x}_4\bar{x}_5 \oplus \bar{x}_3\bar{x}_5 \oplus \bar{x}_3\bar{x}_4 \oplus \bar{x}_3\bar{x}_4\bar{x}_5 \oplus \bar{x}_2\bar{x}_5 \oplus \bar{x}_2\bar{x}_4 \oplus \bar{x}_2\bar{x}_4\bar{x}_5 \oplus \bar{x}_1\bar{x}_3 \oplus$$
$$\bar{x}_1\bar{x}_3\bar{x}_4\bar{x}_5 \oplus \bar{x}_1\bar{x}_2 \oplus \bar{x}_1\bar{x}_2\bar{x}_4\bar{x}_5 \oplus \bar{x}_1\bar{x}_2\bar{x}_3 \oplus x_1x_2x_3x_4x_5.$$

Our computer experiments show that $L_E^{\&}(f) = 13$. Let us remark that $L_E^{\&}(f) > L_E^{\&}(p_5)$ but $L_K^{\&}(f) < L_K^{\&}(p_5)$ and $L_{FK}^{\&}(f) < L_{FK}^{\&}(p_5)$.

As far as we know, Extended class has been not yet discussed in the literature.

3.6 Other Classes

$G(\mathfrak{a})$-classes

Let $\mathfrak{a}$ be an operator. The class of all one-generated bundles by the operator $\mathfrak{a}$ is called $\mathfrak{a}$-*generalized* and is denoted by $G(\mathfrak{a})$.

G-class

The class of all one-generated bundles is called *generalized* and is denoted by G.

Let us remark that the classes $G(\mathsf{d}...\mathsf{d})$ contain all GRM forms [1].

T-class

On the operators $\mathfrak{a} = a_1...a_n$ and $\mathfrak{b} = b_1...b_n$ we define an operation "$\circ$" as follows:

$$\mathfrak{a} \circ \mathfrak{b} = \{1, \; if \; a_i \neq b_i \; for \; all \; i; 0, \; otherwise.$$

For a bundle of operators $\mathsf{A} = (\mathfrak{a}^{\tilde{0}}, ..., \mathfrak{a}^{\tilde{\tau}}, ..., \mathfrak{a}^{\tilde{1}})$ we define *matrix* M_A as follow:

$$M_A = \begin{pmatrix} \mathfrak{a}^{\tilde{1}} \circ \mathfrak{a}^{\tilde{0}} & \cdots & \mathfrak{a}^{\tilde{1}} \circ \mathfrak{a}^{\tilde{\tau}} & \cdots & \mathfrak{a}^{\tilde{1}} \circ \mathfrak{a}^{\tilde{1}} \\ \cdots & & \cdots & & \cdots \\ \mathfrak{a}^{\tilde{0}} \circ \mathfrak{a}^{\tilde{0}} & \cdots & \mathfrak{a}^{\tilde{0}} \circ \mathfrak{a}^{\tilde{\tau}} & \cdots & \mathfrak{a}^{\tilde{0}} \circ \mathfrak{a}^{\tilde{1}} \end{pmatrix}.$$

A bundle A belongs to the class T iff M_A has triangle form. It follow that A is a base bundle.

Remark. All well-known classes of ESOP belong T

OF-class

The class of all base bundles is denoted by OF.

Theorem 3.6 Let ESOP$(f) = S_1 \oplus \ldots \oplus S_m$ be a minimal ESOP of a function f. Then there exists a base bundle $\mathsf{T} = (\mathfrak{t}^{\tilde{0}}, ..., \mathfrak{t}^{\tilde{\tau}}, ..., \mathfrak{t}^{\tilde{1}})$ such that for any S_i there exist a vector $\tilde{\tau}$ such that $S_i = \mathfrak{t}^{\tilde{\tau}}(x_1 \cdot ... \cdot x_n)$.

Remark 1. The class of all base bundles with respect to the function $x_1 \cdot ... \cdot x_n$ equal to Inclusive Forms [2] by complexity of representations.

Remark 2. There exists an efficient algorithm to construct a base bundle from minimal ESOP [3].

4 Hierarchy

Each class contains all classes which are below of it on the diagram. For example, *FK* contains K.

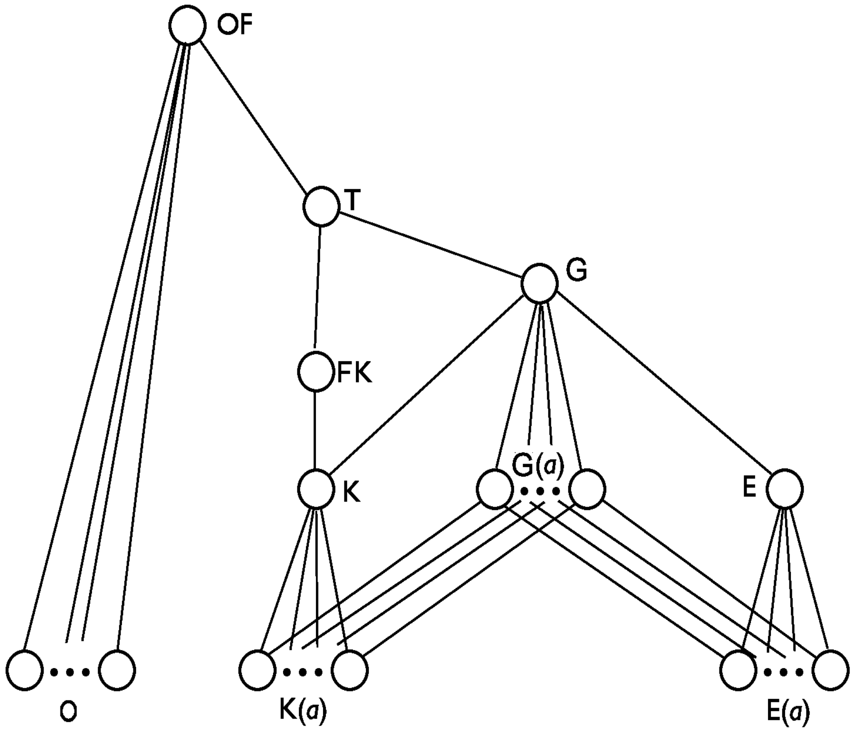

Diagram 1. The hierarchy of the classes of operator forms.

The class $K(\mathfrak{a})$ contains all FPRM when $\mathfrak{a} = \mathfrak{d} \dots \mathfrak{d}$. The class $G(\mathfrak{a})$ contains all GRM when $\mathfrak{a} = \mathfrak{d} \dots \mathfrak{d}$. The class K contains all KRO. The class *FK* contains all FKE. All abbreviations are from [1].

5 Conclusions

There are a lot of classes of operator forms which we didn't show on the diagram 1. We don't have well-represented definitions and non-trivial bounds for the Shannon function for them. All bounds for the Shannon function look like

$$L_R^{\&}(n) \le c \cdot 2^n,$$

where c is a constant. The estimation $L_{OF}^{\&}(n) \le \frac{29}{128} \cdot 2^n$ was obtained in [8] by using a computer. Although, there is an upper bound $L_{OF}^{\&} \le \frac{log_2 n+1}{n} \cdot 2^n$ which was obtained by the theoretical way [6]. It is interested to derive a class R (a sequence of classes R_i) for which the following estimation is true $L_R^{\&}(n) \le c(n)2^n$, where $c(n) \to 0$ as $n \to \infty$.

6 Acknowledgments

This research was partially supported by Grant from Russian Foundation for Basic Research N 01–00–00556.

References

[1] M. Perkowski, L. Jóźwaik, and R. Drechsler. New hierarchies of and/exor trees, decision diagrams, lattice diagrams, canonical forms, and regular layouts // 3rd International Workshop on Applications of the Reed-Muller Expansions in Circuit Design, Sept. 19–20, 1997, Oxford. — Oxford, 1997. — P. 115–132.

[2] M. Chrzanowska-Jeske, M. Perkowski, and A. Mishchenko. How to catch the golden fish of minimum ESOP into the net of canonical forms.

[3] Some questions of the theory of Boolean functions / Ed. N. Peryazev and S. Vinokurov. — M.: FIZMATLIT, 2001. — 192 p. (Russian)
http://www.isu.ru/facs/math/unc/kafedra/diskret/monograph

[4] C. P. Schnorr. Zwei lineare untere Schranken für die Komplexität Boolescher Funktionen // Computing. Archiv für elektronisches Rechnen. — 1974. — V. 13, N 2. — P. 155–171.

[5] A. Gaidukov, S. Vinokurov. Operator polynomial axpansions of Boolean functions // 4^{th} International Workshop on Boolean Problems. Freiberg, Germany, 2000. P. 63–69.

[6] K. Kirichenko. An upper bound on the Shannon function in ESOPs // Optimization Methods and their Applications, 2001, Irkutsk, Baikal. Section 5. Discrete Mathematics. Irkutsk, 2001. P. 66–70. (Russian)

[7] S. Even, I. Kohavi, A. Paz. On minimal modulo 2 sums of products for switching function // IEEE Trans. Elect. Comput., Oct. 1967. P. 671–674.

[8] A. Gaidukov. Algorithm to derive MESOP for 6-variable function // 5^{th} International Workshop on Boolean Problems. Freiberg, Germany, 2002.

[9] A. Al-Rabadi and M. Perkowski. Multiple-Valued Galoas Field S/D Trees for GFSOP Minimization and their Complexity. "Proc. of the International Symposium on Multiple-Valued Logic", Warsaw, Poland, 22–24 May, 2001. P.159–166.