Stability under unanimous consent, free mobility and core*

Anna Bogomolnaia† Michel Le Breton‡ Alexei Savvateev§ Shlomo Weber¶

May 18, 2006

Revised version

Abstract

In this paper we consider an urban population represented by a continuum of individuals uniformly distributed over the real line that faces a problem of location and financing of multiple public facilities. We examine three notions of stability of emerging jurisdiction structures under unanimous consent, free mobility and core, and provide a characterization of stable partitions. We also show that our stability notions may yield stable partitions with sharply distinct jurisdiction sizes.

Keywords: Jurisdiction structures, admission under unanimous consent, equal share, core, free mobility.

JEL Classification Numbers: D70, H20, H73.

---

*We wish to thank an anonymous referee for useful suggestions.

†Rice University, Houston, USA.

‡Université de Toulouse I, GREMAQ and IDEI, Toulouse, France.

§CORE, Catholic University of Louvain, Louvain-la-Neuve, Belgium; New Economic School, Moscow; Central Economics and Mathematics Institute, Moscow; and Institute for Theoretical and Experimental Physics, Moscow. Financial support through grants R98-0631 from the Economic Education and Research Consortium, # NSh-1939.2003.6 School Support, Russian Foundation for Basic Research No. 04-02-17227, and the Russian Science Support Foundation is gratefully acknowledged.

¶Southern Methodist University, Dallas, USA, and CORE, Catholic University of Louvain, Louvain-la-Neuve, Belgium.

1 Introduction

Consider an urban population represented by a continuum of individuals uniformly distributed over the real line. The population faces a problem of locating and financing of one or more public facilities. The group solution for this problem consists of three elements: *jurisdiction structure*, which is a partition of individuals into groups assigned to the same facility; *facility location* in each jurisdiction, and *sharing rule* that determines individual contributions towards the facility cost in each jurisdiction.

In this paper we focus on the search for a stable partition of the entire population into several jurisdictions.[1] In doing so, we impose the principles of *efficiency* and *equal share*. The efficiency requires that a location of the facility in each jurisdiction is chosen in order to minimize the total transportation cost of its residents. Since we assume that the transportation cost of each individual is given by the distance from the facility location, one can show that the efficiency is attained through the majority voting, and each jurisdiction places the facility at the location of its *median* resident. As in Jéhiel and Scotchmer (1997, 2001), Alesina and Spolaore (1997), Casella (2001), Haimanko et al. (2005), Bogomolnaia et al. (2005b), we impose the assumption of equal share, where all members of the same jurisdiction make equal contributions towards the facility cost.[2]

We then introduce several notions of stability that are immune to a possibility of groups of individuals migrating to one of the existing jurisdictions or creating a new one. It is important to stress that, while migrating between jurisdictions, all individuals anticipate the median location of the public facility and the equal share cost mechanism in a newly created jurisdiction. The most demanding migration requirement we consider is what Jéhiel and Scotchmer (2001) call *stability admission under unanimous consent (SAUC)*. This notion grants every individual the veto power

[1]Throughout the paper, we will use the terms *jurisdiction structure* and *partition* interchangeably.

[2]See Haeringer (2000), Le Breton and Weber (2003), Haimanko et al. (2004), Le Breton et al. (2004), Bogomolnaia et al. (2005a) for alternative approaches to cost sharing mechanisms, and Le Breton and Weber (2004) for a general review of cost sharing schemes in this context.

regarding a possible migration of any group of individuals from other jurisdiction to her own. Obviously, the admission under unanimous consent severely restricts threats to stability, thus, generating a large set of stable jurisdiction structures. We then turn to the examination of more permissive stability threats. One is *core stability (CS)* where every group of individuals is allowed to leave their jurisdictions and to create a new one. Another is *stability under free mobility (SFM)* that does not allow the members of existing jurisdiction to prevent migration by members of other jurisdictions. While both CS and SFM notions are stronger than SAUC, we also examine a less obvious link between CS and SFM structures and show that every heterogenous SFM is also core stable. Even though this conclusion does not hold in the case where all jurisdictions are represented by intervals of the same length, one can claim that, in general, the free mobility is the most permissive threat to stability.

An examination of stable partitions has to deal with the questions of number, size and composition of jurisdictions they contain. In characterizing the number of jurisdictions in a stable partition one has to take into account the conflict between increasing returns to scale that favor the creation of larger groups and the heterogeneity of individuals' locations that support the emergence of smaller groups. To discuss the size and the composition of jurisdictions in a stable partition, note that the presence of a sufficient number of distant individuals in the jurisdiction may adversely impact the value of a total jurisdictional transportation cost to the chosen facility location. From this point of view, the locational heterogeneity could be costly and, for a given size of jurisdiction, the intra-heterogeneity is minimal when the jurisdiction is an interval. In local public finance, this property is referred to as *stratification* while in game theory it is often called *consecutiveness.* Another important feature of jurisdiction structures is inter-heterogeneity of jurisdiction sizes, or so-called *heterogeneity gap* in sizes of jurisdictions in stable partitions.

The existence of stable structures for the uniform distribution of players[3] has been studied by Alesina and Spolaore (1997) whose stability analysis yields partitions that consist of equal-size

[3]See Haeringer (2000), Le Breton and Weber (2003), Haimanko et al. (2004) and Bogomolnaia et al. (2005b) for examination of stable structures for non-uniform distributions.

jurisdictions.[4] Obviously, in the case of non-uniform distributions the heterogeneity of jurisdiction sizes is a natural feature of the model. The novelty of our conclusions is that even the uniform distribution of individuals' locations may yield stable structures with sharply distinct jurisdiction sizes.[5] Our results offer an interesting link with Jéhiel and Scotchmer (1997, 2001) who pointed out the existence of stable partitions with heterogeneous sizes in the model where public goods are differentiated according to a single vertical dimension (quantity) and individuals are characterized by their willingness to pay.

The paper is organized as follows. Section 2 describes the model and introduces the notions of stability examined in this paper. In Section 3 we identify jurisdiction structures that satisfy consecutiveness, border indifference, size monotonicity and homogeneity, and examine whether those properties are consistent with the stability notions introduced in previous section. In particular, we show that all SFM partitions are consecutive, whereas it is not necessarily the case for SAUC and CS partitions. In Section 4 we characterize SFM, as well as consecutive SAUC and CS partitions. More specifically, we establish bounds on the size of jurisdictions that yield stable partitions (under either SAUC, SFM or CS). It turns out that stable structures could display a strong size heterogeneity among formed jurisdictions. We also establish a somewhat surprising link between SFM and CS (while both are weaker than SAUC). Namely, every SFM partition that contains jurisdictions of different sizes is also CS. However, the situation is reverse for partitions that consist of equal-size intervals for which every CS partition is SFM. The proofs of all results are relegated to the Appendix B. This is preceded by Appendix A that contains preliminary results and remarks.

2 The Model

We consider a society which faces the problem of location, financing and assignment of its members to public facilities (hospitals, schools, libraries, etc.). For that purpose, society may

[4]See Casella (2001) for a similar conclusion in somewhat different framework.

[5]See also Cechlárová et al. (2001), who, in the discrete framework, examine a much weaker notion of stable structures immune against individual migration and moves by two adjacent individuals from neighboring jurisdictions.

remain as a whole or be partitioned into several jurisdictions. Each jurisdiction selects the location for the facility (not necessarily within jurisdiction's bounds) and finances the cost of this facility by collecting tax from the jurisdiction members. Each individual therefore incurs two costs: the tax (her monetary contribution towards the costs of local facility), and the transportation cost from the individual's own location to that of the facility. The benefit of using the service is assumed to exceed any potential cost, so no individual would stay away from public facilities ensuring the voluntarily participation of all individuals in the process described above.

We assume that the society consists of individuals uniformly distributed over the interval $I = [0, 1]$. Any measurable set $S \subset I$ of a positive Lebesgue measure (not necessarily an interval!) could be an admissible jurisdiction. We call *jurisdiction structure* a partition $P = \{S_i\}_{1 \leq i \leq n}$ of I into a finite number of jurisdictions. We refer to P with n elements as an n-partition. Slightly abusing the notation, we identify an individual with her location, so that we will use just t for an individual located at the point $t \in I$.

The cost of a facility $g > 0$ is independent of location and jurisdiction and is divided equally among jurisdiction members. The residents cover the cost of the facility so that the tax imposed on every member of the jurisdiction S is $\frac{g}{|S|}$, where $|S|$ denotes the measure of jurisdiction S. Residents of each jurisdiction face an idiosyncratic cost represented by the distance to the facility chosen by that jurisdiction: an individual $t \in S$ faces the transportation cost $|t - r|$, if the facility is located at point r.

The efficiency condition implies that every jurisdiction S locates the facility at its median point $m(S)$, which minimizes the total transportation cost of its members (see Haimanko et al. (2004)). If a median point is not unique (which can happen when S is not a connected set), then there is an interval of median points and we assume that $m(S)$ is the midpoint of this interval.

Given the assumptions above, the total cost $c(t, S)$ of an individual t in jurisdiction S is uniquely given by:

$$c(t, S) = |t - m(S)| + \frac{g}{|S|}.$$

For any jurisdiction structure $P = \{S_i\}_{i=1}^n$ and any individual $t \in I$ we denote by S^t the (unique) jurisdiction from P which contains t. We will use the notation $c(t, P)$ for $c(t, S^t)$, the total cost an individual t incurs in the jurisdiction structure P.

An arbitrarily chosen partition of the society could be prone to migrations by some dissatisfied groups (measurable sets) of individuals, who, in search for a better payoff (lower total cost) will switch to another jurisdiction, or even to form a new jurisdiction. Our goal is to identify *stable* partitions, immune to such migrations. We consider various notions of stability, steaming from three different principles for permissible group deviations (or migrations).

The first notion allows a group of individuals S to join an existing jurisdiction T whenever all migrants *and* all members of the migration target T would benefit from the migration move. This notion of stability is called *stability under admission by unanimous consent* (Jéhiel and Scotchmer (2001)):

Definition 2.1: A partition P is *stable under admission by unanimous consent (SAUC)* if there exists no group $S \subset I$ and a jurisdiction $T \in P$, such that $c(t, S \cup T) < c(t, P)$ for all $t \in S \cup T$.

In particular, this implies that no set of measure zero S is allowed to move under SAUC condition. Indeed, in this case the migration move does not impact the members of T, thus violating the strict inequality in the definition. Thus, only a group of a positive measure could present a migration threat under SAUC.

The second notion of stability emerges when a group of individuals is allowed to form a new jurisdiction, as long as all the migrants become better of. This leads to the traditional core stability (CS) notion:

Definition 2.2: A partition P is called *core stable (CS)* if there exists no group $S \subset I$ such that $c(t, S) < c(t, P)$ for every $t \in S$.

Since an admissible under SAUC migration is also an admissible deviation under CS, SAUC is a weaker requirement then CS.

The third possibility is to allow a group of individuals S to join another jurisdiction T when all members of S would be better off regardless of the changes incurred by members of T. The corresponding notion of stability is called *stability under free mobility (SFM).*

Definition 2.3: A partition P is *stable under free mobility (SFM)* if there exists no group $S \in I$, together with a jurisdiction $T \in P$, such that $c(t, S \cup T) < c(t, P)$ for all $t \in S$.

Since SFM does not demand a consent of members of the host jurisdiction, SMF is stronger than SAUC. The relation between SFM and CS is less obvious. Contrary to the CS requirement, under SFM the potential migrants have to join an existing jurisdiction and they are not allowed to form a new one. Nevertheless we will show that, except for the special homogenous case of partitions that consist of equal intervals, CS is implied by SFM. Thus, in general, CS requirement is weaker than SFM. Even more surprisingly, in the homogenous case, the situation is reversed and SFM is implied by CS.

Note also, that unlike in SAUC, both CS and SFM allow for migrant sets of measure zero, or even for deviating individuals.[6]

To summarize the type of admissible deviations, under all three stability notions, a necessary condition for a group S to consider a migration is the strict reduction of the after-migration relative to the pre-migration costs for all members of S. Moreover

• Under SAUC, a deviating group S should join an existing jurisdiction T and make all members of T better off;

• Under CS, a deviating group S should form its own new jurisdiction;

• Under SFM, a deviating group S should join an existing jurisdiction, regardless of cost implications on members of the host jurisdiction.

We now turn to identification of special classes of jurisdiction partitions and their compatibility with stability notions defined in this section.

[6] A threat of migration by sets of null measure under CS is merely hypothetical as the monetary contribution by every individual in such a set is, infact, infinite.

3 Classes of partitions

So far we have not imposed any *ex ante* restrictions on the set of admissible partitions or jurisdictions. Nevertheless, some special types of partitions play important role in potential applications, and will be of particular interest to our analysis. We then investigate a possibility that the restrictions on partitions we impose are consistent with the stability notions we examine. The most important type of partitions are those consisting of *consecutive* (see Greenberg and Weber (1986)) jurisdictions, which are represented by intervals in I (which may or may not contain their endpoints):

Definition 3.1: A partition P of I is *consecutive* if every $S \in P$ is an interval.

A consecutive partition P can be given by the sequence of intervals $\{(x_{i-1}, x_i)\}_{1 \leq i \leq n}$, where $0 = x_0 < x_1 < \ldots x_{n-1} < x_n = 1$. We will denote the length of an interval $S_i = (x_{i-1}, x_i)$ in P by s_i. Given that $x_0 = 0$ and $x_n = 1$, a consecutive partition is uniquely defined by the $n-1$-tuple $(x_1, \ldots, x_{n-1})$, where $0 < x_1 < \ldots x_{n-1} < 1$ or by positive numbers $s_1, \ldots, s_n$, such that $\sum_{i=1}^{n} s_i = 1$. If no confusion will arise, we use the notation $P = (x_1, \ldots, x_{n-1})$ or $P = (s_1, \ldots, s_n)$.

The following property of consecutive partitions, called *border indifference*, would be important for the analysis of a potential migration of "almost peripheral" individuals close to the border with other jurisdictions. It states that a *peripheral* individual, located on the border of two adjacent intervals in P, namely at points $x_1, \ldots, x_{n-1}$, is indifferent between membership in any of these two jurisdictions:

Definition 3.2: A consecutive partition $P = (x_1, \ldots, x_{n-1})$ satisfies *border indifference (BI)* if for all $i = 1, \ldots, n-1$ we have $c(x_i, [x_{i-1}, x_i]) = c(x_i, [x_i, x_{i+1}])$.

Note that a partition P satisfies BI if and only if the function $c(\cdot, P)$ is continuous on I.

We will also consider a subset of consecutive partitions in which the jurisdiction sizes either (weakly) increase or decrease from one endpoint of I to another:

Definition 3.3: A consecutive partition P of I is *size-monotone* if either $s_1 \leq s_2 \leq \ldots \leq s_n$ or $s_1 \geq s_2 \geq \ldots \geq s_n$. (Without loss of generality, we will conduct our analysis for the former case).

If the inequalities in Definition 3.3 turn into equalities, the partitions with equal-size jurisdictions will arise:

Definition 3.4: A partition P of I is *homogenous* if $s_1 = \ldots = s_n$. A non-homogeneous partition (not necessarily size-monotone) will be referred to as *heterogenous partition.*

We now establish the link between properties of partitions introduced in this section and our stability notions. First,

Proposition 3.5: Any SFM partition is consecutive.

However, this conclusion does not hold for other two stability notions we consider. Moreover, even a consecutive CS (and SAUC) partition may not be size-monotone (and thus, not homogeneous):

Proposition 3.6: (i) A CS (and hence a SAUC) partition is not necessarily consecutive.

(ii) A consecutive CS, as well as SFM, partition is not necessarily size-monotone.

In the next section we provide our characterization results for stable partitions.

4 Stable Partitions: Characterization Results

Before proceeding with the characterization of stable consecutive partitions, it would be useful to have a more detailed examination of tax burden imposed on the members of a jurisdiction in the partition. Indeed, consider an interval $S = [a, b]$ of length s and note that the peripheral individuals a and b incur the highest total cost within the jurisdiction.[7] Their total burden is given by the value of the "peripheral" cost function Ψ defined on $\Re_+$:

$$\Psi(s) = \frac{s}{2} + \frac{g}{s}.$$

[7] In the case of closed intervals, two jurisdictions have a common peripheral individual, that could be considered to be a member of both. We discuss this technical point in the Appendix A.

It is easy to see that Ψ is strictly convex, attains its minimum at $d^*(g) = \sqrt{2g}$, and $\min_{s\geq 0} \Psi(s) = \Psi(d^*(g)) = d^*(g)$. Moreover, it decreases for $s < d^*(g)$, increases for $s > d^*(g)$, and takes any value $d > d^*(g)$ twice (once for some $s < d^*(g)$, and once for some $s' > d^*(g)$). Thus, $s = d^*(g)$ is the "optimal size" of an interval jurisdiction that minimizes the cost of the most disadvantaged individuals. Note also that the border indifference condition BI can be presented as $\Psi(s_1) = \ldots = \Psi(s_n)$. Hence, any partition, which satisfies BI, contains jurisdictions of at most two distinct sizes.

We now provide the complete characterization of SFM partitions. In the multi-jurisdictional case we determine the bounds for jurisdictional sizes in a SFM partition:

Proposition 4.1: (i) A consecutive heterogeneous partition P into multiple jurisdictions is SFM if and only if it satisfies BI, and for each $S \in P$ we have $|S| \in \left[\frac{d^*(g)}{\sqrt{2}}, \sqrt{2}d^*(g)\right]$;

(ii) A homogenous partition P into multiple interval jurisdictions of size s is SFM if and only if $s \geq \frac{d^*(g)}{\sqrt{2}}$;

(iii) The grand jurisdiction I is always SFM.

It is worthwhile to establish the link between SFM and CS (to recall both notions are stronger than SAUC). It turns out that in the heterogenous case, SFM is stronger than CS. It is surprising that the situation is reversed in the homogenous case, where the set of CS partitions is a subset of the SFM set. These conclusions are presented by the following:

Proposition 4.2: (i) Every heterogenous SFM partition is CS.

(ii) Every consecutive homogenous CS partition is SFM.

We now turn to the examination of SAUC and CS partitions. Even though Proposition 3.6 indicates that SAUC and CS partitions could be non-consecutive, in what follows we concentrate on the case when only admissible jurisdictions are intervals. Proposition 3.6 also indicates that if a CS (SAUC) partition is consecutive, it need not be size monotone. However, once the size monotonicity is imposed,[8] we obtain sufficient conditions for SAUC and CS.

[8]To recall, we restrict our examination to the case $s_1 \leq \ldots \leq s_n$.

Proposition 4.3: Let $P = (s_1, \ldots, s_n)$ be a size-monotone partition of I where $d^*(g) \leq s_1$. Then P is SAUC.

Proposition 4.4: Let $P = (s_1, \ldots, s_n)$ be a size-monotone partition of I where $d^*(g) \leq s_1 \leq s_n \leq \sqrt{2}d^*(g)$. Then P is CS.

To guarantee that a size-monotone partition is SAUC, it is enough to verify that jurisdictions in this partition are not too small: the size of the smallest jurisdiction should not be lower than the "optimal" size $d^*(g)$. To sustain a stricter requirement of CS, the size of the largest jurisdiction should have an upper bound of $\sqrt{2}d^*(g)$.

If we further restrict our attention to homogenous partitions, we obtain a necessary and sufficient condition for CS:

Proposition 4.5: (i) A homogenous partition of I which consists of multiple jurisdictions of size s, is CS if and only if $\frac{d^*(g)}{\sqrt{2}} \leq s \leq d^*(g)\left(1 + \sqrt{2}\right)$.

(ii) The interval I is CS if and only if[9] $\sqrt{g} \geq \frac{1}{2 + \sqrt{6}}$.

Since $s = \frac{1}{n}$ and $d^*(g) = \sqrt{2g}$, the last proposition determines the values of n which, for given $g > 0$, admit CS n-partitions. Namely,

if $\sqrt{g} \geq \frac{1}{2 + \sqrt{6}}$, then 1-partition is CS;

if $\sqrt{g} \leq \frac{1}{n} \leq \sqrt{g}\left(2 + \sqrt{2}\right)$, where $n > 1$, then there exists a CS n-partition.

We also provide an alternative version of Proposition 4.5 by identifying, for given n, the range of values of g that yield a homogenous CS n-partition.

Proposition 4.6: (i) A multi-jurisdictional homogenous n-partition is CS if and only if

$$\frac{1}{n^2[2 + \sqrt{2}]^2} \leq g \leq \frac{1}{n^2}.$$

(ii) A homogenous CS partition exists for any $g > 0$.

[9] Note that this condition is also necessary and sufficient for (core) stability of a *given* jurisdiction $S_i \in P$ against threats from inside, i.e., against potentially seceding jurisdictions $S \subset S_i$.

5 Appendix A — Preliminary Results

We start by introducing some notation, lemmas and remarks which will be helpful to prove our results. For every jurisdiction S (either a member of an initial jurisdiction structure, or a potentially deviating group), denote by $l(S) = \inf\{t|t \in S\}$ and $r(S) = \sup\{t|t \in S\}$ its peripheral individuals. In order to avoid making presentation unnecessarily complicated by technical details, we will assume that for every S the peripheral individuals $l(S)$ and $r(S)$ belong to S. Then some individuals may belong to two jurisdictions but we chose to ignore a finite number of points (at most n as the number of jurisdictions in the partition).[10]

Let us now extend the notation $c(t, S)$ to the case when an individual $t \notin S$. In this case, we write $c(t, S)$ for the total cost $c(t, S \cup \{t\})$ an individual t would incur if she joins jurisdiction S. Note that, given S, $c(t, S) = c(t, S \cup \{t\}) = |t - m(S)| + \dfrac{g}{|S|}$ is a continuous single-dipped function. For any partition P, any jurisdiction S, and any individual $t \in I$, we define $\Delta(t, S, P) = c(t, S) - c(t, P)$. Recall that, whenever a group S is allowed to deviate from P (under either SAUC, CS or SFM), all members of S must be strictly better off relatively to their cost levels in P. Hence, S can deviate only if $\Delta(t, S, P) < 0$ for all $t \in S$.

We will use the following remarks and lemmas. Let $g > 0$ be given. We will write d^* instead of $d^*(g)$ for the optimal size of jurisdiction.

Remark A.1. The peripheral cost function $\Psi(s) = \dfrac{s}{2} + \dfrac{g}{s}$, defined in Section 4, satisfies the following conditions:

$$\frac{d\Psi(s)}{ds} \in \left[-\frac{1}{2}, \frac{1}{2}\right] \quad \text{for } s \in [\sqrt{g}, +\infty).$$

$$\Psi(s + a) > \Psi(s) - \frac{|a|}{2} \quad \text{for} \quad s \in [\sqrt{g}, +\infty) \quad \text{and } a > -s.$$

This follows from direct differentiation and the fact that the convexity of Ψ implies $\Psi(s + a) > \Psi(s) + \Psi'(s)a$.

Remark A.2. If P is SAUC, then for any $S_i, S_j \in P$, $i \neq j$ we have $m(S_i) \neq m(S_j)$.

[10] All the statements in the paper remain true without this assumption and the proofs are available from authors upon request.

Straightforward, since otherwise group S_i would join group S_j, making everyone in $S_i \bigcup S_j$ better off.

Remark A.3. For any S we have $\min\{c(l(S), S), c(r(S), S)\} \geq \Psi(|S|) \geq d^*$ and $\max\{c(l(S), S), c(r(S), S)\} = \max\{c(t, S) : t \in S\} \geq \Psi(|r(S) - l(S)|)$. All inequalities are straightforward except the last one. Assume that $c(l(S), S) \geq c(r(S), S)$. Then $m(S) \geq m([l(S), r(S)])$, and

$$c(l(S), S) = |l(S) - m(S)| + \frac{g}{|S|} \geq |l(S) - m([l(S), r(S)])| + \frac{g}{|r(S) - l(S)|} = \Psi(|r(S) - l(S)|).$$

Remark A.4. If t is a peripheral individual in S, then $c(t, S) \geq \min_s \Psi(s) = d^* = \sqrt{2g} > 1.4\sqrt{g}$. If S is a group that violates SAUC of the jurisdiction structure P, then $c(t, P) > d^*$ (hence, if S^t is an interval, it must be that $|S^t| \neq d^*$).

Remark A.5. Let $P = (s_1, \ldots, s_n)$ be a consecutive partition, with $s_i \in [\sqrt{g}, 2\sqrt{g}]$. Then for any $t \in I$ we have

$$c(t, P) \leq \max_{1 \leq i \leq n} \{\Psi(s_i)\} \leq \max\{\Psi(s) : s \in [\sqrt{g}, 2\sqrt{g}]\} = \Psi(\sqrt{g}) = \Psi(2\sqrt{g}) = 1.5\sqrt{g}.$$

Lemma A.6. Let $P = (x_1, \ldots, x_{n-1})$ be a consecutive partition, with $s_i \in [\sqrt{g}, 2\sqrt{g}]$. If a group S violates CS of P, then there exist i, j such that $\max\{|l(S) - x_i|, |r(S) - x_j|\} < 0.1\sqrt{g}$.

Proof: We can assume $l(S) \in S_i = [x_{i-1}, x_i] \in P$ and $l(S) \geq m(S_i)$. Then

$$c(x_i, S_i) - c(l(S), P) = c(x_i, S_i) - c(l(S), S_i) = |x_i - m(S_i)| - |l(S) - m(S_i)| = |l(S) - x_i|.$$

Since S violates core stability of P, Remarks A.5 and A.4 imply that $c(x_i, S_i) \leq 1.5\sqrt{g}$ and $c(l(S), P) > c(l(S), S) > 1.4\sqrt{g}$. Then the conclusion follows immediately. □

Lemma A.7. Let P be a consecutive partition, satisfying BI. Then for every jurisdiction S, the function $\Delta(\cdot, S, P)$ is a continuous (in fact, a peace-wise linear) function on I, single-dipped, and attains its minimum at $m(S)$.

Proof: Continuity follows from BI. For $t \leq m(S)$ we have

$$\Delta(t, S, P) \quad = \quad c(t, S) - c(t, P) = |t - m(S)| - |t - m(S^t)| + \frac{g}{|S|} - \frac{g}{|S^t|}$$

$$= \quad m(S) - t - |t - m(S^t)| + C_1(S^t) = \begin{cases} C_2(S^t), & t \le m\left(S^t\right); \\ C_2(S^t) - 2t, & t \ge m\left(S^t\right), \end{cases}$$

where both functions $C_1(S^t)$ and $C_2(S^t)$ are constant on each S^t. Hence, $\Delta(t, S, P)$ is (non-strictly) decreasing on each interval $S^t \cap [0, m(S)]$. Thus, it is decreasing on $[0, m(S)]$. Analogously, it is increasing on $[m(S), 1]$. $\square$

Lemma A.8. Let P be a consecutive partition, satisfying BI. If no interval can violate CS of P, then P is CS.

Proof: Assume that S violates CS of P by deviating from P and making all its members better off. Then $\Delta(l(S), S, P) < 0$ and $\Delta(r(S), S, P) < 0$. Hence, by Lemma A.7, $\Delta(t, S, P) < 0$ on the entire interval $[l(S), r(S)]$. But then the interval $[m(S) - \frac{|S|}{2}, m(S) + \frac{|S|}{2}]$ also makes all its members better off by deviating from P, a contradiction. $\square$

We now turn to the proof of our main results.

6 Appendix B — Main Results

Proof of Proposition 3.5: Assume to the contrary, that P is a nonconsecutive SFM. Then there are distinct $S, S' \in P$ and individuals $t_1, t_2 \in S$, $t' \in S'$, such that $t_1 < t' < t_2$. Without loss of generality, we can assume $m(S) < m(S')$ (see Remark A.2), and that $t_1 < m(S) < t_2$. Then for all $t \in I$ we have:

$$c(t, S) - c(t, S') = |t - m(S)| - |t - m(S')| + \frac{g}{|S|} - \frac{g}{|S'|}$$

$$= |t - m(S)| - |t - m(S')| + C = \begin{cases} m(S) - m(S') + C, & t \le m(S); \\ 2t - m(S) - m(S') + C, & m(S) \le t \le m(S'); \\ m(S') - m(S) + C, & m(S') \le t. \end{cases}$$

Hence, $c(t, S) - c(t, S')$ is an increasing function of t on $[0, 1]$, and for $t' \in (m(S), m(S'))$ we have $c(t_1, S) - c(t_1, S') < c(t', S) - c(t', S') < c(t_2, S) - c(t_2, S')$. But SFM implies that no individual can be better off changing her jurisdiction. In particular, we have $c(t_1, S) \le c(t_1, S')$, $c(t_2, S) \le c(t_2, S')$, and $c(t', S') \le c(t', S)$, which contradicts the above inequalities. If $t' \notin (m(S), m(S'))$, then the detailed proof is quite involved, and is available from the authors, upon the request. □

Proof of Proposition 3.6: (i) Let $g = \frac{1}{18}$ and therefore $d^* = \frac{1}{3}$. Consider P that consists of three jurisdictions, $S_1 = [0, d^* - \varepsilon] \bigcup [1-\varepsilon, 1]$, $S_2 = (d^* - \varepsilon, 2d^* - \varepsilon)$, $S_3 = (2d^* - \varepsilon, 1 - \varepsilon)$. Assume, in negation, that a jurisdiction S can deviate from P and violate CS of P. By Remark A.4, neither $l(S)$ nor $r(S)$ can belong to either S_2, S_3 or $[0, d^* - \varepsilon]$, since any individual t located in one of those areas has $c(t, P) \leq d^*$. Hence, $S \subset [1 - \varepsilon, 1] \subset S_1$.

Note that for any $t \in I$ we have $c(t,P) = |t - m(S^t)| + \dfrac{g}{d^*} \leq 1 + \dfrac{g}{d^*} = \dfrac{7}{6}$. So, if S deviates, it has to be $c(t,S) < c(t,P) \leq \dfrac{7}{6}$ for all $t \in S$. But, for ε small enough, $c(t,S) \geq \dfrac{g}{|S|} \geq \dfrac{1}{18\varepsilon} \geq \dfrac{7}{6}$, a contradiction.

(ii) Let $g = \frac{1}{36}$ and $d^* = \frac{\sqrt{2}}{6}$. Consider a consecutive 4-partition $P = (s_1, s_2, s_3, s_4)$, where $s_1 = s_4 = \frac{1}{6}$ and $s_2 = s_3 = \frac{1}{3}$. By Proposition 4.1, part (i), it is SFM, and by Proposition 4.2, part (i), it is CS, too. However, P is obviously not size-monotone. □

Proof of Proposition 4.1 (i)"Only if" part: Let $P = \{S_1, \ldots, S_n\} = (x_1, \ldots, x_{n-1}) = (s_1, \ldots, s_n)$ be a consecutive heterogeneous SFM n-partition, $n > 1$.

Assume that BI condition does not hold. Then, without loss of generality, there exists an individual x_i, $i \in \{1, \ldots, n-1\}$, peripheral for S_i and S_{i+1}, such that $c(x_i, S_i) > c(x_i, S_{i+1})$. But then for any small enough positive ε we have $c(x_i - \varepsilon, S_i) > c(x_i - \varepsilon, S_{i+1})$, so the interval $(x_i - \varepsilon, x_i)$ would benefit from migrating to S_{i+1}, which contradicts SFM.

Now, since P satisfies BI and is heterogeneous, it contains jurisdictions of exactly two sizes, s' and s'', where $s' < s''$ and $\Psi(s') = \Psi(s'')$. To complete this part, it suffices to verify that $s' \geq \sqrt{g}$ (since $\Psi(\sqrt{g}) = \Psi(2\sqrt{g})$, the inequality $s'' \leq 2\sqrt{g}$ then follows).

Assume, to the contrary, that $s' < \sqrt{g}$. Then we have $s'' > 2\sqrt{g} > 2s'$ and $s' < \dfrac{g}{s'}$. Let, without loss of generality, $S_1 = [0, s']$ and $S_2 = [s', s' + s'']$. Choose ε such that $0 < \varepsilon < (s'' - 2s')/2$ and

$\varepsilon < \Psi(s'') - \Psi(2s' + \varepsilon)$ (such $\varepsilon > 0$ exists, since $\Psi(s'') > \Psi(2s')$ and Ψ is continuous). We show that the group $T = [s'' - \varepsilon, s' + s''] \subset S_2$ would benefit from joining $S_1 = [0, s']$.

First, note that $|T| = s' + \varepsilon > s'$, so $m(T \cup S_1) \in T$; moreover, $m(T \cup S_1) = s'' - \frac{\varepsilon}{2} > s'' - \varepsilon > m(S_2) = s' + \frac{s''}{2}$. We will now check that $\Delta(t, T \cup S_1, P) = c(t, T \cup S_1) - c(t, P) < 0$ for all $t \in T$. First,

$$\Delta(s' + s'', T \cup S_1, P) = c(s' + s'', T \cup S_1) - c(s' + s'', P) = \Psi(2s' + \varepsilon) - \Psi(s'') < -\varepsilon < 0.$$

Next, for $t \in T$ with $m(S_2) < m(T \cup S_1) = s'' - \frac{\varepsilon}{2} \leq t \leq s' + s''$, we have

$$\Delta(t, T \cup S_1, P) = c(t, T \cup S_1) - c(t, P) = \Delta(s' + s'', T \cup S_1, P) = \Psi(2s' + \varepsilon) - \Psi(s'') < -\varepsilon < 0.$$

Finally, for $t \in T$ with $m(S_2) < s'' - \varepsilon \leq t \leq s'' - \frac{\varepsilon}{2} = m(T \cup S_1)$, we have

$$\Delta(t, T \cup S_1, P) = \Delta(s' + s'', T \cup S_1, P) + 2(m(T \cup S_1) - t) =$$

$$\Psi(2s' + \varepsilon) - \Psi(s'') + (2s'' - \varepsilon - 2t) < 2(s'' - \varepsilon - t) \leq 0.$$

Thus, T would benefit from deviation, which contradicts SFM. $\square$

Proof of Proposition 4.1 (ii). "Only if" part: Let P be a homogenous partition into intervals of size s. If $s < \sqrt{g}$, then $s < \frac{g}{s}$, and a jurisdiction $S_1 = [0, s]$ would benefit from joining $S_2 = [s, 2s]$. Indeed, for any $t \in S_1$ we have

$$c(t, S_1) = |t - \frac{s}{2}| + \frac{g}{s} \geq |t - s| - |\frac{s}{2} - s| + \frac{g}{s} = |t - s| + \frac{g}{s} - \frac{s}{2} > |t - s| + \frac{g}{2s} = c(t, S_1 \cup S_2).$$

Note that in this case all members of $S_1 \cup S_2$ benefit form being together. Thus, for $s < \sqrt{g}$ partition P is neither SFM nor CS. $\square$

We will use Proposition 4.2 to prove the "if" part of Proposition 4.1 (i) and (ii), so we prove Proposition 4.2 before completing the proof of Proposition 4.1.

Proof of Proposition 4.2: The "only if" part of Proposition 4.1 (i) implies that if a heterogeneous P is SFM then it satisfies BI and consists of intervals of two different sizes, s' and s'', where $\sqrt{g} \le s < s'' \le 2\sqrt{g}$ and $\Psi(s') = \Psi(s'') = \psi \le 1.5\sqrt{g}$. Proposition 4.2 follows from the stronger statement:

Proposition B.1: If a consecutive partition P satisfies BI and $\sqrt{g} \le |S_i| \le 2\sqrt{g}$ for all jurisdictions $S_i \in P$, then P is CS.

Proof: If P satisfies the conditions of Proposition B.1, then it satisfies BI and hence consists of intervals of at most two sizes, s' and s'', where $\sqrt{g} \le s' \le s'' \le 2\sqrt{g}$ and $\Psi(s') = \Psi(s'') = \psi \le 1.5\sqrt{g}$.

Hence, by Remark A.5, we have $c(t, P) \le 1.5\sqrt{g}$ for any $t \in P$. If a jurisdiction S can deviate under CS, then by Lemma A.8 we may assume it is an interval. Remark A.3 then yields $\Psi(|S|) = c(t', S) < c(t', P) \le 1.5\sqrt{g}$, where t' is a peripheral individual in S. Thus, $|S| \in (\sqrt{g}, 2\sqrt{g})$.

By Lemma A.6 we can find x_i and x_j, peripheral individuals in jurisdictions in P, such that that both differences $a_1 = |l(S) - x_i|$ and $a_2 = |r(S) - x_j|$ are smaller than $0.1\sqrt{g}$. Without loss of generality, assume that $a_1 \ge a_2$. We have $S = [l(S), r(S)] = [x_i \pm a_1, x_j \pm a_2]$ and $i \le j$. We now consider the following four possible cases.

CASE 1: $i = j$. Here $|S| = r(S) - l(S) \le |l(S) - x_i| + |r(S) - x_i| \le 0.2\sqrt{g}$, which contradicts $|S| \in (\sqrt{g}, 2\sqrt{g})$. Hence, this is impossible.

CASE 2: $i = j - 1$, so $[x_{j-1}, x_j] = S_j \in P$. Then, given $a_1 \ge a_2$, and Remark A.1:

$$c(l(S), S) = \Psi(|S|) = \Psi(s_j \pm a_1 \pm a_2) > \Psi(s_j) - \frac{| \pm a_1 \pm a_2|}{2} \ge \psi - |a_1| = c(l(S), P),$$

a contradiction to S being a deviating group.

CASE 3: $i = j - 2$. Then, since Ψ is increasing on $[\sqrt{2g}, +\infty)$, we have:

$$c(l(S), S) = \Psi(|S|) = \Psi(s_j + s_{j-1} \pm a_1 \pm a_2) > \Psi(s_j + s_{j-1}) - \frac{| \pm a_1 \pm a_2|}{2} \ge$$

$$\Psi(2\sqrt{g}) - |a_1| = 1.5\sqrt{g} - |a_1| \ge \psi - |a_1| = c(l(S), P),$$

a contradiction. Finally,

CASE 4: $j - i \geq 3$. Then, $|S| \geq 3\sqrt{g} - 0.2\sqrt{g} = 2.8\sqrt{g} \notin (\sqrt{g}, 2\sqrt{g})$, again a contradiction, which completes the proof of both Proposition B.1 and part (i) of Proposition 4.2. Part (ii) of the latter proposition follows immediately from Proposition 4.5 and (already proved) the "only if" part of Proposition 4.1. Part (ii). $\square$

Proof of Proposition 4.1 (i). "If" part. Again, we will prove a bit stronger statement, namely:

Proposition B.2: If a consecutive partition P satisfies BI and $\sqrt{g} \leq |S_i| \leq 2\sqrt{g}$ for all jurisdictions $S_i \in P$, then P is SFM.

Proof: Consider a consecutive P, which satisfies BI. Hence it consists of intervals of at most two sizes, s' and s'', where, by the assumption, $\sqrt{g} \leq s' \leq s'' \leq 2\sqrt{g}$ and so $\Psi(s') = \Psi(s'') = \psi \leq 1.5\sqrt{g}$. Proposition B.1 guarantees that P is CS.

Assume that P is not SFM. Then there exists a group T, and a jurisdiction $S_i \in P$ such that all members of T prefer $T \cup S_i$ to T. We can choose T so that $T \cap S_i = \emptyset$. Partition T into T_l and T_r, which lie respectively to the left and to the right of $S_i = [x_{i-1}, x_i]$ so that $t \leq x_{i-1}$ for all $t \in T_l$ and $t \geq x_i$ for all $t \in T_r$.

First, one of T_l and T_r should be empty. Indeed, let $T' = T \cup S_i$ with $l(T') \in T_l$, $r(T') \in T_r$. Since both values $\Delta(l(T'), T', P)$ and $\Delta(r(T'), T', P)$ are negative, Lemma A.7 implies that $\Delta(t, T', P) < 0$ for all $t \in [l(T'), r(T')]$. But this means that the group T' can deviate by forming its own jurisdiction, a contradiction to the fact that P is CS.

Now, without loss of generality, assume $T_r = \emptyset$ and $T = T_l$. By Remark A.5 we have $1.5\sqrt{g} \geq c(l(S), P) > c(l(S), T') \geq \Psi(|T'|)$, and so $|T'| < 2\sqrt{g}$. Since $|S_i| \geq \sqrt{g}$, we obtain $|T| < \sqrt{g}$ and so $m(T') \in S_i$.

Next, note that the individual located at $m(T')$ is better off at T' than at S_i: her tax contribution declines since jurisdiction becomes larger, and her transportation cost drops to zero. Thus, $\Delta(l(T'), T', P) < 0$, $\Delta(m(T'), T', P) < 0$, and Lemma A.7 implies that $\Delta(t, T', P) < 0$ for all $t \in [l(T'), m(T')]$. Then the interval $T'' = [x_{i-1} - |T|, x_{i-1}]$, with $|T''| = |T| < \sqrt{g}$, also can deviate under SFM by joining the adjacent interval S_i. Denote $p = x_{i-1} - |T|$, so that $T'' = [p, x_{i-1}]$.

If $|S_i| = s''$, then $c(p, S_i \cup T'') = \Psi(|S_i \cup T''|) > \Psi(s'') = \Psi(s') \geq c(p, P)$, which contradicts the assumption that T'' can deviate under SFM. Hence, $|S_i| = s'$.

Furthermore, we have $|T''| < \sqrt{g}$, and its left endpoint, p belongs to S_{i-1}. From Remark A.1 we obtain:

if $p \geq m(S_{i-1})$, then

$$c(p, T'' \cup S_i) = \Psi\left(|T'' \cup S_i|\right) = \Psi(s' + |T''|) > \Psi(s') - |T''| = \psi - |T''| = c(p, S_{i-1}) = c(p, P);$$

if $p \leq m(S_{i-1})$, then

$$c(p, T'' \cup S_i) = \Psi\left(|T'' \cup S_i|\right) = \Psi(s' + s_{i-1} - (s_{i-1} - |T|)) > \Psi(s' + s_{i-1}) - (s_{i-1} - |T|) \geq$$

$$\Psi(2\sqrt{g}) - (s_{i-1} - |T|) > \psi - (s_{i-1} - |T|) = c(p, S_{i-1}) = c(p, P).$$

Hence, both possibilities contradict the assumption that T'' can deviate by joining S_i under SFM. This completes the proof of both Proposition B.2 and Proposition 4.1 (i). □

Proof of Proposition 4.1 (ii). "If" part: Let P be a homogenous partition into intervals of size $s \geq \sqrt{g}$. If $\sqrt{g} \leq s \leq 2\sqrt{g}$, then SFM follows from Proposition B.2 above. Let $s \geq 2\sqrt{g}$. Suppose that a group T joins a jurisdiction $S_i = [(i-1)s, is] \in P$. Denote $T \cup S_i$ by S. Either $l(S)$ or $r(S)$ belong to T, assume it is $l(S)$. By Remark A.1, we have

$$c(l(S), S) \geq \Psi(|S|) \geq \Psi(s) \geq c(l(S), P),$$

and the individual at $l(S)$ would not benefit from this migration. Thus, there are no profitable migrations, and P is SFM. □

Proposition 4.1 (iii) is obvious.

Proof of Proposition 4.3. Let $P = \{S_1, \ldots, S_n\} = (s_1, \ldots, s_n)$ be a size-monotone partition, with $d = \sqrt{2g} \le s_1 \le \ldots \le s_n$. If it is not SAUC, then there exists $T \subset I$ and $S_i \in P$, such that $S = T \cup S_i$ is preferred to P by all its members.

Let S be such that $l(S) \in S_j \in P$, $j \le i$. Using Remarks A.1 and A.3, we obtain $c(l(S), P) = c(l(S), S_j) \le \Psi(s_j) \le \Psi(s_i) < \Psi(|S|) \le c(l(S), S)$, a contradiction. □

Proof of Proposition 4.4: Let $P = \{S_1, \ldots, S_n\} = (s_1, \ldots, s_n) = (x_1, \ldots, x_{n-1})$ be a size-monotone partition, with $d = \sqrt{2g} \le s_1 \le \ldots \le s_n \le 2\sqrt{g}$. By Remark A.5, $c(t, P) \le 1.5\sqrt{g}$ for all $t \in I$. If P is not CS, then there exists a group $T \subset I$ which would be better of by forming its own jurisdiction. By Lemma A.6, there exist i and j, such that both differences $a_1 = |l(S) - x_i|$ and $a_2 = |r(S) - x_j|$ are smaller than $0.1\sqrt{g}$.

Since $s_k \ge \sqrt{2g}$ for all $S_k \in P$, we have $i \le j$.
If $i = j$, then $|T| < 0.2\sqrt{g}$ and $c(l(S), T) \ge \dfrac{g}{|T|} > 1.5\sqrt{g} \ge c(l(S), P)$.
If $i < j - 1$, then $|T| > 1.8\sqrt{2g}$ and $c(l(S), T) > \Psi(1.8\sqrt{2g}) > 1.5\sqrt{g} \ge c(l(S), P)$.

Both cases contradict the fact that T can beneficially deviate under CS, hence it must be $i = j - 1$. This means that the individuals $x_i = x_{j-1}$ and x_j are peripherals of $S_j \in P$. Thus, $r(S) - l(S) \ge s_j - 0.2\sqrt{g} \ge \sqrt{2g} - 0.2\sqrt{g} = (\sqrt{2} - 0.2)\sqrt{g}$.

Consider two possible cases:

CASE 1: $|T| < |S_j| = s_j$. Then $c(m(S_j), P) < c(m(S_j), T)$ (the contribution of $m(S_j)$ is larger in T, while her transportation cost is smaller (zero) at P), hence $m(S_j) \notin T$. For the same reason, no individual in the part of S_j located between $m(S_j)$ and its peripheral individual, which does not contain $m(T)$ in its interior, cannot belong to T. This observation yields $|T| \le \dfrac{s_j}{2} + 0.2\sqrt{g}$.

Moreover,

$$\max\{c(l(S),T),c(r(S),T)\} \geq \frac{|r(S)-l(S)|}{2} + \frac{g}{|T|} \geq \frac{1}{2}\left(s_j - 0.2\sqrt{g}\right) + \frac{g}{\frac{s_j}{2}+0.2\sqrt{g}} =$$

$$\left(\frac{s_j}{4} - 0.2\sqrt{g}\right) + \frac{1}{2}\left(\frac{s_j}{2}+0.2\sqrt{g}\right) + \frac{g}{\frac{s_j}{2}+0.2\sqrt{g}} = \left(\frac{s_j}{4} - 0.2\sqrt{g}\right) + \Psi\left(\frac{s_j}{2}+0.2\sqrt{g}\right).$$

Since $\sqrt{2g} \leq s_j \leq 2\sqrt{g}$, we have $\frac{s_j}{2} + 0.2\sqrt{g} \leq 1.2\sqrt{g} < d^*$, and, given that Ψ decreases on $[0, d^*]$, we obtain $\Psi\left(\frac{s_j}{2}+0.2\sqrt{g}\right) \geq \Psi(1.2\sqrt{g})$. Thus, we can continue:

$$\max\{c(l(S),T),c(r(S),T)\} \geq \left(\tfrac{\sqrt{2g}}{4} - 0.2\sqrt{g}\right) + \Psi(1.2\sqrt{g}) = \left(\tfrac{\sqrt{2}}{4} - 0.2\right)\sqrt{g} + 0.6\sqrt{g} + \tfrac{1}{1.2}\sqrt{g}$$

$$= \left(\tfrac{\sqrt{2}}{4} + 0.4 + \tfrac{5}{6}\right)\sqrt{g} = \tfrac{15\sqrt{2}+74}{60}\sqrt{g} > 1.5\sqrt{g} = \max_{t\in I} c(t,P).$$

This contradicts the fact that both $l(S)$ and $r(S)$ joins the deviating group T.

CASE 2: $|T| \geq |S_j| = s_j$. Since $|l(S) - x_{j-1}| \leq 0.1\sqrt{g}$, we have $l(S) \in S_j$ or $l(S) \in S_{j-1}$, so $\sqrt{2g} \leq |S^{l(S)}| \leq |T|$. Hence, $c(l(S),T) \geq \Psi(|T|) \geq \Psi(S^{l(S)}) = c(l(S), S^{l(S)}) = c(l(S), P)$, so the individual $l(S)$ does not improve from P to T. This contradiction completes the proof of the proposition. □

Proof of Proposition 4.5: Since any homogenous consecutive partition satisfies BI, Lemma A.8 applies, so P is CS if no interval can deviate.

Part (i), $n > 1$**; "only if".** We have already checked (see the proof of Proposition 4.1 (ii), "only if" part), that for $s < \sqrt{g}$, P can not be CS.

If $s > (2+\sqrt{2})\sqrt{g}$, then $T = \left[x_1 - \frac{\sqrt{g}}{2}, x_1 + \frac{\sqrt{g}}{2}\right]$, the interval of the size $\sqrt{g}$, whose median is located at one of the peripheral points of P would benefit from deviation. Indeed, among the members of T, the individual $x_1 - \frac{\sqrt{g}}{2}$ has the highest cost in T and the lowest cost in P, but still

$$c(x_1 - \frac{\sqrt{g}}{2}, P) = \Psi(s) - \frac{\sqrt{g}}{2} > 2\sqrt{g} - \frac{\sqrt{g}}{2} \geq 1.5\sqrt{g} = c(x_1 - \frac{\sqrt{g}}{2}, T).$$

Part (i), $n > 1$**; "if".** If $s \in [\sqrt{g}, 2\sqrt{g}]$, then Proposition B.1 (utilized in the proof of Proposition 4.2 above) guarantees that P is CS. Let $s \in (2\sqrt{g}, [2+\sqrt{2}]\sqrt{g}]$, so that $1.5\sqrt{g} \leq \Psi(s) \leq 2\sqrt{g}$, and let $S = [l(S), r(S)]$ be a deviating interval.

Similarly to Lemma A.6, there exist peripheral (relative to jurisdictions in P) individuals x_i and x_j, such that $\max\{|l(S)-x_i|, |r(S)-x_j|\} < 0.6\sqrt{g}$. Indeed, without loss of generality, assume that $l(S) \in S_i = [x_{i-1}, x_i] \in P$, $l(S) \geq m(S_i)$. Then $|l(S) - x_i| = c(x_i, S_i) - c(l(S), S_i) = \Psi(s) - c(l(S), P) < \Psi(s) - c(l(S), S) < 2\sqrt{g} - 1.4\sqrt{g} = 0.6\sqrt{g}$.

Now, since $2\sqrt{g} \leq s$, we have $i \geq j$.

The case $i = j$ is similar to the proof of Proposition 4.4, and is skipped. If $i < j$, then we can assume, without loss of generality, that $a_1 = |l(S) - x_i| \geq a_2 = |r(S) - x_j|$. Using Remark A.1, we again obtain a contradiction of the beneficial deviation of T:

$$c(l(S), T) = \Psi(|T|) = \Psi((j-i)s \pm a_1 \pm a_2) > \Psi(s) - \frac{|\pm a_1 \pm a_2|}{2} \geq \Psi(s) - |a_1| = c(l(S), P).$$

This completes the proof of part (i).

Part (ii), $n = 1$**.** Let $P = \{I\}$, and assume that all members of $T = [l(S), r(S)] \in I$ benefit from its deviation. Without loss of generality, let $m(T) < m(I) = \frac{1}{2}$. First, if $r(S) \geq \frac{1}{2}$, then individual $r(S)$ would be worse off at T than at I (both contribution and transportation cost would increase). Hence, $r(S) < \frac{1}{2}$.

Then $T' = [0, p]$, where $p = r(S) - l(S)$, also can deviate under CS. Indeed, for every individual $t \in T' = [0, p]$ we have $c(t, T') = c(t + l(S), T) < c(t + l(S), I) \leq c(t, I)$. If T' can deviate under CS, then since $p < \frac{1}{2}$ and all members of T' pay tax of at least $2g$, we have $2g < c(t, T') < c(t, I) \leq \frac{1}{2} + g$, which implies $g < \frac{1}{2}$. Hence, for $g \geq \frac{1}{2}$ partition $I = \{I\}$ will be CS.

Further, interval $T' = [0, p]$ can deviate if and only if $\Delta(t, T', \{I\}) < 0$ for all $t \in T'$. But

$$\Delta(t, T', \{I\}) = c(t, T') - c(t, I) = \begin{cases} \Psi(p) - \Psi(1), & 0 \leq t \leq \frac{p}{2} \\ \Psi(p) - \Psi(1) + (2t - p), & \frac{p}{2} \leq t \leq p \end{cases}$$

is increasing on $[0, p]$, hence, T' can deviate if and only if $\Delta(p, T', \{I\}) < 0$. Now

$$\Delta(p, T', \{I\}) = c(p, T') - c(p, I) = \left(\frac{p}{2} + \frac{g}{p}\right) - \left(\frac{1}{2} - p + g\right) = -\left(\frac{1}{2} + g\right) + \left(\frac{3}{2}p + \frac{g}{p}\right).$$

For given g, this function is a convex in p, and attains its minimum at $p^* = \sqrt{2g/3}$. But the range of p is $[0, \frac{1}{2}]$; hence, for $g \geq \frac{3}{8}$ this function has minimum in the corner point $p = \frac{1}{2}$, where it is obviously positive; hence, core stability is assured for $g \geq \frac{3}{8}$, and we analyse the case $g < \frac{3}{8}$ in what follows.

In this range for g, there exists an interval $[0, p]$ that can profitably deviate if and only if $\Delta(p^*, [0, p^*], \{I\}) < 0$. Finally,

$$\Delta(p^*, [0, p^*], \{I\}) = -\left(\frac{1}{2} + g\right) + \left(\frac{3}{2}p^* + \frac{g}{p^*}\right) = \sqrt{6g} - \left(\frac{1}{2} + g\right).$$

Thus, $P = \{I\}$ is not CS if and only if $g < \dfrac{3}{8}$ and $\sqrt{6g} - g - \dfrac{1}{2} < 0 \Leftrightarrow \sqrt{g} < \dfrac{1}{2 + \sqrt{6}}$. $\square$

Proposition 4.6 (i) is the immediate consequence of Proposition 4.5, and Proposition 4.6, part (ii) follows from part (i) by simple calculations. $\square$

7 References

Alesina, A. and E. Spolaore (1997), On the number and size of nations, *Quarterly Journal of Economics* 112, 1027-1056.

Bogomolnaia, A., Le Breton, M., Savvateev, A. and S. Weber (2005a), The egalitarian sharing rule in provision of public projects, *Economics Bulletin* 8 (11), 1-5.

Bogomolnaia, A., Le Breton, M., Savvateev, A. and S. Weber (2005b), Stability of jurisdiction structures under the equal share and median rules, CORE Discussion Paper.

Cassela, A. (2001), The role of market size in the formation of jurisdictions, *Review of Economic Studies* 68, 83-108.

Cechl*á*rov*á*, K, Dahm, M. and V. Lacko (2001) “Efficiency and stability in a discrete model of country formation”, *Journal of Global Optimization* 20, 239-256.

Greenberg, J. and S. Weber (1986), Strong Tiebout equilibrium under restricted preferences domain, *Journal of Economic Theory* 38, 101-117.

Jéhiel, P. and S. Scotchmer (1997), Free mobility and the optimal number of jurisdictions, *Annals d’Economie et Statistiques* 45, 219-231.

Jéhiel, P. and S. Scotchmer (2001), Constitutional rules of exclusion in jurisdiction formation, *Review of Economic Studies* 68, 393-413.

Haeringer, G. (2000) “Stable coalition structures with fixed decision scheme”, Discussion paper.

Haimanko, O., Le Breton, M. and S. Weber (2004), Voluntary formation of communities for provision of public projects, *Journal of Economic Theory* 115, 1-34.

Haimanko, O., Le Breton, M. and S. Weber (2005), Transfers in a polarized country: bridging the gap between efficiency and stability, *Journal of Public Economics*, 89, 1277-1303.

Le Breton, M. and S. Weber (2003), The art of making everybody happy: how to prevent a secession, *IMF Stuff Papers*, 50, 403-435.

Le Breton, M. and S. Weber (2004), Secession-proof cost allocations and stable group structures in models of horizontal differentiation, in *Group Formation in Economics: Networks, Clubs and Coalitions*, Demange, G. and M. Wooders, eds., Cambridge University Press, 266-285.

Le Breton, M., Weber, S. and J.H. Drèze (2004), The Rawlsian principle and secession-proofness in large heterogeneous societies, CORE Discussion Paper.