Analysis of Deterministic Cash Flows and the Term Structure of Interest Rates

Cash Flow

Financial transactions and investment opportunities are described by cash flows they generate. Cash flow: payment made or received. Multiple cash flows are called *cash flow streams*.

Time Value of Money

A dollar today is worth more than a dollar tomorrow. A dollar today can be invested to start earning *interest* immediately.

Valuation Problem

Find ***present value*** of all future cash flows related to a financial transaction or investment opportunity.

Two types of valuation problems:

- ***Deterministic*** *valuation problems* (the timing and the amount of all cash flows are known with certainty on the valuation date)
- ***Stochastic*** valuation problems (the timing and/or the amount of future cash flows are not known with certainty at the valuation date)

Simple Interest

Simple interest: the interest accrued I on the loan with notional A is calculated as

$$I = r \cdot \tau \cdot A$$

where τ is the time elapsed (in the fraction of the year), r is interest rate. τ is calculated according to one of the day count conventions. For example, when day count convention is **actual/actual**:

$\tau = \frac{n}{N}$, n is number of days elapsed, N is the total number of days in the year.

Usually, simple interest is used for short periods (less than one year).

Compound Interest

Compound interest: each interest payment is reinvested to earn more interest in subsequent periods. Compound interest with annual compounding: start with the amount of money A . After n years you will have $(1+r)^n A$:

$$A \to (1+r)A \to (1+r)^2 A \to \dots \to (1+r)^n A$$

Present Value Calculations with Annual Compounding Frequency

- Present value of a certain cash flow one year from now:

$$PV = \frac{A}{1+r}$$

- Present value of a certain cash flow two years from now:

$$PV = \frac{A}{(1+r)^2}$$

- Present value of a certain cash flow n years from now:

$$PV = \frac{A}{(1+r)^n}$$

The Additivity Property of Present Values

- Present values are expressed in current dollars, so they are additive.
- Valuing a stream of future cash flows occurring at ***different times*** in the future – discounted cash flow (DCF) formula:

$$PV = \frac{\sum_{n=1}^{N} C_n}{(1+r)^n}$$

where C_n is the cash flow at the end of year n.

Two deterministic cash-flow streams are called **equivalent** if their present values are equal.

Compounding Intervals, Compounding Frequency, and Continuous Compounding

- Previously we considered annual compounding. Any compounding interval can be considered.
- **Example:**
  - A 10% interest rate compounded **semiannually** implies two six-month 5% interest periods per year. A $100 investment will be worth $105 after six months and $110.25 after one year (the $105 earns 5%).
  - This produces higher return than the 10% interest rate with **annual** compounding (the $100 investment invested for one year at 10% per annum with annual compounding will be worth $110 at the end of one year).
- Generally, an investment of A at the rate of r_m per year compounded m times per year by the end of nth year amounts to:

$$\left(1+\frac{r_m}{m}\right)^{mn} A$$

- **Example:** valuing a coupon-bearing bond with coupons paid m times per year (a whole number of coupons remaining until maturity)

  PV(Bond) = PV(coupon stream) + PV(principal), i.e.,

$$PV = \frac{cF}{m} \sum_{k=1}^{N} \frac{1}{(1+r/m)^k} + \frac{F}{(1+r/m)^N},$$

where

- c : annual coupon rate (e.g., 6% per year or 0.06)
- F : face value of the bond (e.g., $1,000)
- m : coupon payment and compounding frequency (e.g., semiannual coupon, i.e., coupons are paid twice a year and m = 2)
- N : number of whole coupon periods between today and maturity (N/m – number of years remaining to maturity)

Continuous Compounding

- Continuous compounding is obtained in the limit when the compounding interval becomes infinitesimally small and compounding frequency goes to infinity $m \to \infty$.

$$\lim_{m \to \infty} \left(1+\frac{r_m}{m}\right)^{mn} = e^{r \cdot n}$$

- For practical purposes, continuous compounding often approximates *daily* compounding. Continuous compounding is very important as it is used in derivatives pricing.
    - **Compounding:** future value at the end of nth year of the amount A invested today for n years: $e^{r\,n}A$.
    - **Discounting:** present value of A to be received in n years: $e^{-r\,n}A$.

- **Relationships between continuous compounding and compounding with frequency m times per year .**
    - Suppose that r_c is the rate of interest with continuous compounding and r_m is the equivalent rate of interest with compounding m times per annum. Then we have the relationship:

$$\left(1+\frac{r_m}{m}\right)^m = e^{r_c}$$

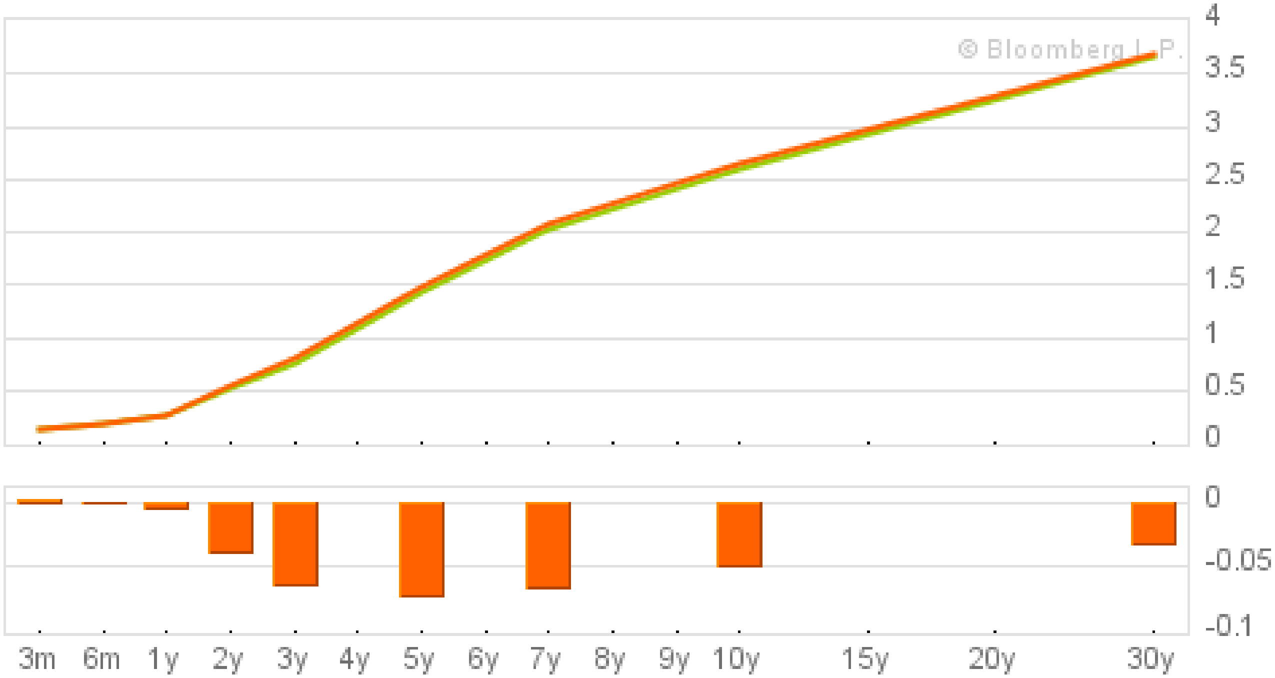
- **Example:** Future value of $1,000 at the end of one year invested at 10% per annum, where the interest rate is quoted with different compounding frequencies:

Annual (m=1)	1,100.00
Semiannual (m=2)	1,102.50
Quarterly (m=4)	1,103.80
Monthly (m=12)	1,104.71
Daily (m=365)	1,105.16
Continuous	1,105.17

This example shows that whenever an interest rate is quoted to you, you should always find out what compounding frequency is used to produce the quote!

The Term Structure of Interest Rates

Above we assumed that the interest rate for all maturities is the same. In reality, there is a term structure of interest rates. Here is the term structure on 30 of August of 2010 (from Bloomberg):



Zero-coupon Bonds

- A default-free zero-coupon bond with the face value F maturing at time T in the future is a security that pays F at time T (investor receives one certain cash flow F at time T).
- Zero-coupon bond prices are quoted as a percent of par (face value).
- Prices of zero-coupon bonds maturing at different times and having face values of one dollar are **basic building blocks** to value other deterministic cash flows.
- *Notation:* $\boldsymbol{P(t,T)}$ denotes the price at time t of a zero-coupon bond maturing at time T and having the face value of one dollar.

Coupon-bearing Bonds

Consider a coupon-bearing bond maturing at time T in the future, with the principal (face value) F, and coupons C_i at times t_i, $i=1,2,\ldots,N$. Then the coupon-bearing bond can be represented as a portfolio of zero-coupon bonds (zeros). The bond price at time zero is:

$$PV = \sum_{i=1}^{N} P(0,t_i)C_i + P(0,t_N)F$$

Zero-coupon bond yields and spot rates

$$R(t,T) = -\frac{1}{T-t}\ln(P(t,T))$$

The yield $R(t,T)$ as a function of maturity T (when t is fixed) is called the zero-coupon **yield curve** at time t. It shows the relationship between zero-coupon rates and maturity T. Zero-coupon yields are also called **spot rates**.

Day Count Convention and Accrued Interest

- So far we have worked with coupon-bearing bonds assuming there are a whole number of coupons left between today (the valuation date) and maturity. What if we are purchasing a bond on some date that falls between the two coupon payment dates? Then we need to perform what is known as the **accrued interest calculation**.
- *Accrued interest* represents the value of interest earned (accrued) since the last coupon payment date.
- **Reference period** = time between two coupon payments.
- **Day count convention**: X/Y,
  - where X: defines the way in which the number of days between two dates is counted;
  - Y: defines the way in which the total number of days in the reference period is computed.
- The following three types of day count conventions are used in practice:
  1. **Actual/actual** (in period)
  2. **30/360**
  3. **Actual/360**

- **In the U.S. fixed income markets:**
    - Actual/actual is used for T-bonds (semiannual coupons)
    - 30/360 is used for corporate and municipal bonds (semiannual coupons)
    - Actual/360 is used for T-bills and money market instruments
- Interest earned between two dates = (interest earned in the reference period) * (number of days between dates) / (number of days in the reference period).

- **Example:** calculate accrued interest on an 8% bond since the last coupon payment.
    - *Coupons:* March 1 - September 1;
      Last coupon: March 1;
      Today's date: July 3.
    - Calculate accrued interest from the last coupon on March 1:
        - March 1 - September 1: **184** actual days
          180 days on the 30 day basis
        - March 1 - July 3: **124** actual days
          122 = 4*30 + 2 days on the 30 day basis

- Then the accrued interest is:

  4*124/184 = $2.6957 on the actual/actual basis for T-bonds, or

  4*122/180 = $2.7111 on the 30/360 basis for corporate bonds.

- The bond prices reported in the financial pages do not include the accrued interest (they are called clean prices). The price the buyer actually pays (dirty price) is the quoted (clean) price plus the accrued interest

- **Example: YTM for corporate bond (semiannual compounding)**

  Consider the corporate bond paying coupons twice a year at the annual rate c . To calculate YTM (the yield-to-maturity) we need to solve the following equation:

$$B+A=\frac{cF}{2}\sum_{k=1}^{N}\frac{1}{(1+r/2)^{k-1+\tau}}+\frac{F}{(1+r/2)^{N-1+\tau}},$$

  where

  - B -quoted (clean price) (without accrued interest)
  - A – accrued interest to be added to the clean price (A + B is the “dirty” price to be paid by the bond buyer)
  - c – annual coupon rate
  - r – yield-to-maturity to be found by solving this equation numerically
  - N – the number of coupons remaining until maturity
  - τ = (Number of days until the next coupon payment)/(Number of days in the coupon period)