

# Colloquium: Statistical mechanics of money, wealth, and income

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This Colloquium reviews statistical models for money, wealth, and income distributions developed in the econophysics literature since the late 1990s. By analogy with the Boltzmann-Gibbs distribution of energy in physics, it is shown that the probability distribution of money is exponential for certain classes of models with interacting economic agents. Alternative scenarios are also reviewed. Data analysis of the empirical distributions of wealth and income reveals a two-class distribution. The majority of the population belongs to the lower class, characterized by the exponential (“thermal”) distribution, whereas a small fraction of the population in the upper class is characterized by the power-law (“superthermal”) distribution. The lower part is very stable, stationary in time, whereas the upper part is highly dynamical and out of equilibrium.

“Money, it’s a gas.” Pink Floyd, *Dark Side of the Moon*

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## I. HISTORICAL INTRODUCTION

This Colloquium article is based on the lectures that one of us (V.M.Y.) has frequently given during the last nine years, when econophysics became a popular subject. Econophysics is a new interdisciplinary research field applying methods of statistical physics to problems in economics and finance. The term “econophysics” was first introduced by the theoretical physicist Eugene Stanley in 1995 at the conference *Dynamics of Complex Systems*, which was held in Kolkata as a satellite meeting to the STATPHYS-19 conference in China (Carbone *et al.*, 2007; Chakrabarti, 2005). The

term appeared first by Stanley *et al.* (1996) in the proceedings of the Kolkata conference. The paper presented a manifesto of the new field, arguing that “behavior of large numbers of humans (as measured, e.g., by economic indices) might conform to analogs of the scaling laws that have proved useful in describing systems composed of large numbers of inanimate objects” (Stanley *et al.*, 1996). Soon the first econophysics conferences were organized: *International Workshop on Econophysics*, Budapest, 1997 and *International Workshop on Econophysics and Statistical Finance*, Palermo, 1998 (Carbone *et al.*, 2007), and the book *An Introduction to Econophysics* by Mantegna and Stanley (1999) was published.

The term econophysics was introduced by analogy with similar terms, such as astrophysics, geophysics, and biophysics, which describe applications of physics to different fields. Particularly important is the parallel with biophysics, which studies living organisms, but they still obey the laws of physics. Econophysics does not literally apply the laws of physics, such as Newton’s laws or quantum mechanics, to humans. It uses mathematical methods developed in statistical physics to study statistical properties of complex economic systems consisting of a large number of humans. As such, it may be considered as a branch of applied theory of probabilities. However, statistical physics is distinctly different from mathematical statistics in its focus, methods, and results.

Originating from physics as a quantitative science, econophysics emphasizes quantitative analysis of large amounts of economic and financial data, which became increasingly available with the introduction of computers and the Internet. Econophysics distances itself from the verbose, narrative, and ideological style of political

economy and is closer to econometrics in its focus. Studying mathematical models of a large number of interacting economic agents, econophysics has much common ground with the agent-based modeling and simulation. Correspondingly, it distances itself from the representative-agent approach of traditional economics, which, by definition, ignores statistical and heterogeneous aspects of the economy.

Another direction related to econophysics has been advocated by the theoretical physicist Serge Galam since early 1980 under the name of sociophysics (Galam, 2004), with the first appearance of the term by Galam *et al.* (1982). It echoes the term “physique sociale” proposed in the nineteenth century by Auguste Comte, the founder of sociology. Unlike econophysics, the term “sociophysics” did not catch on when first introduced, but it is coming back with the popularity of econophysics and active support from some physicists (Schweitzer, 2003; Stauffer, 2004; Weidlich, 2000). While the principles of both fields have much in common, econophysics focuses on the narrower subject of economic behavior of humans, where more quantitative data is available, whereas sociophysics studies a broader range of social issues. The boundary between econophysics and sociophysics is not sharp, and the two fields enjoy a good rapport (Chakrabarti, Chakraborti, and Chatterjee, 2006).

Historically, statistical mechanics was developed in the second half of the nineteenth century by James Clerk Maxwell, Ludwig Boltzmann, and Josiah Willard Gibbs. These physicists believed in the existence of atoms and developed mathematical methods for describing their statistical properties. There are interesting connections between the development of statistical physics and statistics of social phenomena, which were recently highlighted by the science journalist Philip Ball (2002, 2004).

Collection and study of “social numbers”, such as the rates of death, birth, and marriage, has been growing progressively since the seventeenth century (Ball, 2004, Ch. 3). The term “statistics” was introduced in the eighteenth century to denote these studies dealing with the civil “states”, and its practitioners were called “statists”. Popularization of social statistics in the nineteenth century is particularly accredited to the Belgian astronomer Adolphe Quetelet. Before the 1850s, statistics was considered an empirical arm of political economy, but then it started to transform into a general method of quantitative analysis suitable for all disciplines. It stimulated physicists to develop statistical mechanics in the second half of the nineteenth century.

Rudolf Clausius started development of the kinetic theory of gases, but it was James Clerk Maxwell who made a decisive step of deriving the probability distribution of velocities of molecules in a gas. Historical studies show (Ball, 2004, Ch. 3) that, in developing statistical mechanics, Maxwell was strongly influenced and encouraged by the widespread popularity of social statistics at the time

(Gillispie, 1963).<sup>1</sup> This approach was further developed by Ludwig Boltzmann, who was very explicit about its origins (Ball, 2004, p. 69):

“The molecules are like individuals, . . . and the properties of gases only remain unaltered, because the number of these molecules, which on the average have a given state, is constant.”

In his book *Populäre Schriften*, Boltzmann (1905) praises Josiah Willard Gibbs for systematic development of statistical mechanics. Then, Boltzmann says:<sup>2</sup>

“This opens a broad perspective, if we do not only think of mechanical objects. Let’s consider to apply this method to the statistics of living beings, society, sociology and so forth.”

It is worth noting that many now-famous economists were originally educated in physics and engineering. Vilfredo Pareto earned a degree in mathematical sciences and a doctorate in engineering. Working as a civil engineer, he collected statistics demonstrating that distributions of income and wealth in a society follow a power law (Pareto, 1897). He later became a professor of economics at Lausanne, where he replaced Léon Walras, also an engineer by education. The influential American economist Irving Fisher was a student of Gibbs. However, most of the mathematical apparatus transferred to economics from physics was that of Newtonian mechanics and classical thermodynamics (Mirowski, 1989; Smith and Foley, 2008). It culminated in the neoclassical concept of mechanistic equilibrium where the “forces” of supply and demand balance each other. The more general concept of statistical equilibrium largely eluded mainstream economics.

With time, both physics and economics became more formal and rigid in their specializations, and the social origin of statistical physics was forgotten. The situation is well summarized by Philip Ball (Ball, 2004, p. 69):

“Today physicists regard the application of statistical mechanics to social phenomena as a new and risky venture. Few, it seems, recall how the process originated the other way around, in the days when physical science and social science were the twin siblings of a mechanistic philosophy and when it was not in the least disreputable to invoke the habits of people to explain the habits of inanimate particles.”

Some physicists and economists attempted to connect the two disciplines during the twentieth century. Frederick Soddy (1933), the Nobel Prize winner in chemistry

<sup>1</sup> V.M.Y. is grateful to Stephen G. Brush for this reference.

<sup>2</sup> Cited from Boltzmann (2006). V.M.Y. is grateful to Michael E. Fisher for this quote.

for his work on radioactivity, published the book *Wealth, Virtual Wealth and Debt*, where he argued that the real wealth is derived from the energy use in transforming raw materials into goods and services, and not from monetary transactions. He also warned about dangers of excessive debt and related “virtual wealth”, thus anticipating the Great Depression. His ideas were largely ignored at the time, but resonate today (Defilla, 2007). The theoretical physicist Ettore Majorana (1942) argued in favor of applying the laws of statistical physics to social phenomena in a paper published after his mysterious disappearance. The statistical physicist Elliott Montroll co-authored the book *Introduction to Quantitative Aspects of Social Phenomena* (Montroll and Badger, 1974). Several economists (Blume, 1993; Durlauf, 1997; Foley, 1994; Follmer, 1974) applied statistical physics to economic problems. The mathematicians Farjoun and Machover (1983) argued that many paradoxes in classical political economy can be resolved if one adopts a probabilistic approach. An early attempt to bring together the leading theoretical physicists and economists at the Santa Fe Institute was not entirely successful (Anderson, Arrow, and Pines, 1988). However, by the late 1990s, the attempts to apply statistical physics to social phenomena finally coalesced into the robust movements of econophysics and sociophysics.

Current standing of econophysics within the physics and economics communities is mixed. Although an entry on econophysics has appeared in the *New Palgrave Dictionary of Economics* (Rosser, 2008a), it is fair to say that econophysics has not been accepted yet by mainstream economics. Nevertheless, a number of open-minded, nontraditional economists have joined this movement, and the number is growing. Under these circumstances, econophysicists have most of their papers published in physics journals. The journal *Physica A: Statistical Mechanics and its Applications* has emerged as the leader in econophysics publications and has even attracted submissions from some *bona fide* economists. Gradually, reputable economics journals are also starting to publish econophysics papers (Gabaix *et al.*, 2006; Lux and Sornette, 2002; Wyart and Bouchaud, 2007). The mainstream physics community is generally sympathetic to econophysics, although it is not uncommon for econophysics papers to be rejected by *Physical Review Letters* on the grounds that “it is not physics”. There is a PACS number for econophysics, and *Physical Review E* has published many papers on this subject. There are regular conferences on econophysics, such as *Applications of Physics in Financial Analysis* (sponsored by the European Physical Society), *Nikkei Econophysics Symposium*, *Econophysics Colloquium*, and *Econophys-Kolkata* (Chakrabarti, 2005; Chatterjee, Yarlagadda, and Chakrabarti, 2005). Econophysics sessions are included in the annual meetings of physical societies and statistical physics conferences. The overlap with economists is the strongest in the field of agent-based simulation. Not surprisingly, the conference

series WEHIA/ESHIA, which deals with heterogeneous interacting agents, regularly includes sessions on econophysics. More information can be found in the reviews by Farmer, Shubik, and Smith (2005); Samanidou *et al.* (2007) and on the Web portal Econophysics Forum <http://www.unifr.ch/econophysics/>.

## II. STATISTICAL MECHANICS OF MONEY DISTRIBUTION

When modern econophysics started in the middle of 1990s, its attention was primarily focused on analysis of financial markets. Soon after, another direction, closer to economics than finance, has emerged. It studies the probability distributions of money, wealth, and income in a society and overlaps with the long-standing line of research in economics studying inequality in a society.<sup>3</sup> Many papers in the economic literature (Champernowne, 1953; Gibrat, 1931; Kalecki, 1945) use a stochastic process to describe dynamics of individual wealth or income and to derive their probability distributions. One might call this a one-body approach, because wealth and income fluctuations are considered independently for each economic agent. Inspired by Boltzmann’s kinetic theory of collisions in gases, econophysicists introduced an alternative, two-body approach, where agents perform pairwise economic transactions and transfer money from one agent to another. Actually, this approach was pioneered by the sociologist John Angle (1986, 1992, 1993, 1996, 2002) already in the 1980s. However, his work was largely unknown until it was brought to the attention of econophysicists by the economist Thomas Lux (2005). Now, Angle’s work is widely cited in econophysics literature (Angle, 2006). Meanwhile, the physicists Ispolatov, Krapivsky, and Redner (1998) independently introduced a statistical model of pairwise money transfer between economic agents, which is equivalent to the model of Angle. Soon, three influential papers by Bouchaud and Mézard (2000); Chakraborti and Chakrabarti (2000); Drăgulescu and Yakovenko (2000) appeared and generated an expanding wave of follow-up publications. For pedagogical reasons, we start reviewing this subject with the simplest version of the pairwise money transfer models presented in Drăgulescu and Yakovenko (2000). This model is the most closely related to the traditional statistical mechanics, which we briefly review first. Then we discuss the other models mentioned above, as well as numerous follow-up papers.

Interestingly, the study of pairwise money transfer and the resulting statistical distribution of money

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<sup>3</sup> See, e.g., Atkinson and Bourguignon (2000); Atkinson and Piketty (2007); Champernowne (1953); Champernowne and Cowell (1998); Gibrat (1931); Kakwani (1980); Kalecki (1945); Pareto (1897); Piketty and Saez (2003).

has virtually no counterpart in modern economics, so econophysicists initiated a new direction here. Only the search theory of money (Kiyotaki and Wright, 1993) is somewhat related to it. This theory was an inspiration for the early econophysics paper by Bak, Nørrelykke, and Shubik (1999) studying dynamics of money. However, a probability distribution of money among the agents was only recently obtained within the search-theoretical approach by the economist Miguel Molico (2006). His distribution is qualitatively similar to the distributions found by Angle (1986, 1992, 1993, 1996, 2002, 2006) and by Ispolatov, Krapivsky, and Redner (1998), but its functional form is unknown, because it was obtained only numerically.

### A. The Boltzmann-Gibbs distribution of energy

The fundamental law of equilibrium statistical mechanics is the Boltzmann-Gibbs distribution. It states that the probability  $P(\varepsilon)$  of finding a physical system or subsystem in a state with the energy  $\varepsilon$  is given by the exponential function

$$P(\varepsilon) = c e^{-\varepsilon/T}, \quad (1)$$

where  $T$  is the temperature, and  $c$  is a normalizing constant (Wannier, 1987). Here we set the Boltzmann constant  $k_B$  to unity by choosing the energy units for measuring the physical temperature  $T$ . Then, the expectation value of any physical variable  $x$  can be obtained as

$$\langle x \rangle = \frac{\sum_k x_k e^{-\varepsilon_k/T}}{\sum_k e^{-\varepsilon_k/T}}, \quad (2)$$

where the sum is taken over all states of the system. Temperature is equal to the average energy per particle:  $T \sim \langle \varepsilon \rangle$ , up to a numerical coefficient of the order of 1.

Eq. (1) can be derived in different ways (Wannier, 1987). All derivations involve the two main ingredients: statistical character of the system and conservation of energy  $\varepsilon$ . One of the shortest derivations can be summarized as follows. Let us divide the system into two (generally unequal) parts. Then, the total energy is the sum of the parts:  $\varepsilon = \varepsilon_1 + \varepsilon_2$ , whereas the probability is the product of probabilities:  $P(\varepsilon) = P(\varepsilon_1) P(\varepsilon_2)$ . The only solution of these two equations is the exponential function (1).

A more sophisticated derivation, proposed by Boltzmann, uses the concept of entropy. Let us consider  $N$  particles with the total energy  $E$ . Let us divide the energy axis into small intervals (bins) of width  $\Delta\varepsilon$  and count the number of particles  $N_k$  having the energies from  $\varepsilon_k$  to  $\varepsilon_k + \Delta\varepsilon$ . The ratio  $N_k/N = P_k$  gives the probability for a particle to have the energy  $\varepsilon_k$ . Let us now calculate the multiplicity  $W$ , which is the number of permutations of the particles between different energy bins such that the occupation numbers of the bins do

not change. This quantity is given by the combinatorial formula in terms of the factorials

$$W = \frac{N!}{N_1! N_2! N_3! \dots}. \quad (3)$$

The logarithm of multiplicity is called the entropy  $S = \ln W$ . In the limit of large numbers, the entropy per particle can be written in the following form using the Stirling approximation for the factorials

$$\frac{S}{N} = - \sum_k \frac{N_k}{N} \ln \left( \frac{N_k}{N} \right) = - \sum_k P_k \ln P_k. \quad (4)$$

Now we would like to find what distribution of particles among different energy states has the highest entropy, i.e., the highest multiplicity, provided the total energy of the system,  $E = \sum_k N_k \varepsilon_k$ , has a fixed value. Solution of this problem can be easily obtained using the method of Lagrange multipliers (Wannier, 1987), and the answer is given by the exponential distribution (1).

The same result can be also derived from the ergodic theory, which says that the many-body system occupies all possible states of a given total energy with equal probabilities. Then it is straightforward to show (López-Ruiz *et al.*, 2008) that the probability distribution of the energy of an individual particle is given by Eq. (1).

### B. Conservation of money

The derivations outlined in Sec. II.A are very general and only use the statistical character of the system and the conservation of energy. So, one may expect that the exponential Boltzmann-Gibbs distribution (1) would apply to other statistical systems with a conserved quantity.

The economy is a big statistical system with millions of participating agents, so it is a promising target for applications of statistical mechanics. Is there a conserved quantity in the economy? Drăgulescu and Yakovenko (2000) argued that such a conserved quantity is money  $m$ . Indeed, the ordinary economic agents can only receive money from and give money to other agents. They are not permitted to “manufacture” money, e.g., to print dollar bills. Let us consider an economic transaction between agents  $i$  and  $j$ . When the agent  $i$  pays money  $\Delta m$  to the agent  $j$  for some goods or services, the money balances of the agents change as follows

$$\begin{aligned} m_i &\rightarrow m'_i = m_i - \Delta m, \\ m_j &\rightarrow m'_j = m_j + \Delta m. \end{aligned} \quad (5)$$

The total amount of money of the two agents before and after transaction remains the same

$$m_i + m_j = m'_i + m'_j, \quad (6)$$

i.e., there is a local conservation law for money. The rule (5) for the transfer of money is analogous to the transfer of energy from one molecule to another in molecular

collisions in a gas, and Eq. (6) is analogous to conservation of energy in such collisions. Conservative models of this kind are also studied in some economic literature (Kiyotaki and Wright, 1993; Molico, 2006).

We should emphasize that, in the model of Drăgulescu and Yakovenko (2000) [as in the economic models of Kiyotaki and Wright (1993); Molico (2006)], the transfer of money from one agent to another represents payment for goods and services in a market economy. However, the model of Drăgulescu and Yakovenko (2000) only keeps track of money flow, but does not keep track of what goods and service are delivered. One reason for this is that many goods, e.g., food and other supplies, and most services, e.g., getting a haircut or going to a movie, are not tangible and disappear after consumption. Because they are not conserved, and also because they are measured in different physical units, it is not very practical to keep track of them. In contrast, money is measured in the same unit (within a given country with a single currency) and is conserved in local transactions (6), so it is straightforward to keep track of money flow. It is also important to realize that an increase in material production does not produce an automatic increase in money supply. The agents can grow apples on trees, but cannot grow money on trees. Only a central bank has the monopoly of changing the monetary base  $M_b$  (McConnell and Brue, 1996). (Debt and credit issues are discussed separately in Sec. II.D.)

Unlike, ordinary economic agents, a central bank or a central government can inject money into the economy, thus changing the total amount of money in the system. This process is analogous to an influx of energy into a system from external sources, e.g., the Earth receives energy from the Sun. Dealing with these situations, physicists start with an idealization of a closed system in thermal equilibrium and then generalize to an open system subject to an energy flux. As long as the rate of money influx from central sources is slow compared with relaxation processes in the economy and does not cause hyperinflation, the system is in quasi-stationary statistical equilibrium with slowly changing parameters. This situation is analogous to heating a kettle on a gas stove slowly, where the kettle has a well-defined, but slowly increasing, temperature at any moment of time. A flux of money may be also produced by international transfers across the boundaries of a country. This process involves complicated issues of multiple currencies in the world and their exchange rates (McCauley, 2008). Here we use an idealization of a closed economy for a single country with a single currency. Such an idealization is common in economic literature. For example, in the two-volume *Handbook of Monetary Economics* (Friedman and Hahn, 1990), only the last chapter out of 23 chapters deals with an open economy.

Another potential problem with conservation of money is debt. This issue will be discussed in Sec. II.D. As a starting point, Drăgulescu and Yakovenko (2000) considered simple models, where debt is not permitted, which

is also a common idealization in some economic literature (Kiyotaki and Wright, 1993; Molico, 2006). This means that money balances of the agents cannot go below zero:  $m_i \geq 0$  for all  $i$ . Transaction (5) takes place only when an agent has enough money to pay the price:  $m_i \geq \Delta m$ , otherwise the transaction does not take place. If an agent spends all money, the balance drops to zero  $m_i = 0$ , so the agent cannot buy any goods from other agents. However, this agent can still receive money from other agents for delivering goods or services to them. In real life, money balance dropping to zero is not at all unusual for people who live from paycheck to paycheck.

Enforcement of the local conservation law (6) is the key feature for successful functioning of money. If the agents were permitted to “manufacture” money, they would be printing money and buying all goods for nothing, which would be a disaster. The physical medium of money is not essential here, as long as the local conservation law is enforced. The days of gold standard are long gone, so money today is truly the fiat money, declared to be money by the central bank. Money may be in the form of paper currency, but today it is more often represented by digits on computerized bank accounts. The local conservation law (6) is consistent with the fundamental principles of accounting, whether in the single-entry or the double-entry form. More discussion of banks, debt, and credit will be given in Sec. II.D. However, the macroeconomic monetary policy issues, such as money supply and money demand (Friedman and Hahn, 1990), are outside of the scope of this paper. Our goal is to investigate the probability distribution of money among economic agents. For this purpose, it is appropriate to make the simplifying macroeconomic idealizations, as described above, in order to ensure overall stability of the system and existence of statistical equilibrium in the model. The concept of “equilibrium” is a very common idealization in economic literature, even though the real economies might never be in equilibrium. Here we extend this concept to a statistical equilibrium, which is characterized by a stationary probability distribution of money  $P(m)$ , as opposed to a mechanical equilibrium, where the “forces” of demand and supply match.

### C. The Boltzmann-Gibbs distribution of money

Having recognized the principle of local money conservation, Drăgulescu and Yakovenko (2000) argued that the stationary distribution of money  $P(m)$  should be given by the exponential Boltzmann-Gibbs function analogous to Eq. (1)

$$P(m) = c e^{-m/T_m}. \quad (7)$$

Here  $c$  is a normalizing constant, and  $T_m$  is the “money temperature”, which is equal to the average amount of money per agent:  $T = \langle m \rangle = M/N$ , where  $M$  is the total

money, and  $N$  is the number of agents.<sup>4</sup>

To verify this conjecture, Drăgulescu and Yakovenko (2000) performed agent-based computer simulations of money transfers between agents. Initially all agents were given the same amount of money, say, \$1000. Then, a pair of agents  $(i, j)$  was randomly selected, the amount  $\Delta m$  was transferred from one agent to another, and the process was repeated many times. Time evolution of the probability distribution of money  $P(m)$  is illustrated in computer animation videos by Chen and Yakovenko (2007) and by Wright (2007). After a transitory period, money distribution converges to the stationary form shown in Fig. 1. As expected, the distribution is well fitted by the exponential function (7).

Several different rules for  $\Delta m$  were considered by Drăgulescu and Yakovenko (2000). In one model, the transferred amount was fixed to a constant  $\Delta m = \$1$ . Economically, it means that all agents were selling their products for the same price  $\Delta m = \$1$ . Computer animation (Chen and Yakovenko, 2007) shows that the initial distribution of money first broadens to a symmetric Gaussian curve, characteristic for a diffusion process. Then, the distribution starts to pile up around the  $m = 0$  state, which acts as the impenetrable boundary, because of the imposed condition  $m \geq 0$ . As a result,  $P(m)$  becomes skewed (asymmetric) and eventually reaches the stationary exponential shape, as shown in Fig. 1. The boundary at  $m = 0$  is analogous to the ground-state energy in statistical physics. Without this boundary condition, the probability distribution of money would not reach a stationary state. Computer animations (Chen and Yakovenko, 2007; Wright, 2007) also show how the entropy of money distribution, defined as  $S/N = -\sum_k P(m_k) \ln P(m_k)$ , grows from the initial value  $S = 0$ , where all agents have the same money, to the maximal value at the statistical equilibrium.

While the model with  $\Delta m = 1$  is very simple and instructive, it is not realistic, because all prices are taken to be the same. In another model considered by Drăgulescu and Yakovenko (2000),  $\Delta m$  in each transaction is taken to be a random fraction of the average amount of money per agent, i.e.,  $\Delta m = \nu(M/N)$ , where  $\nu$  is a uniformly distributed random number between 0 and 1. The random distribution of  $\Delta m$  is supposed to represent the wide variety of prices for different products in the real economy. It reflects the fact that agents buy and consume many different types of products, some of them simple and cheap, some sophisticated and expensive. Moreover, different agents like to consume these products in different quantities, so there is a variation in the paid amounts  $\Delta m$ , even when the unit price of the same product is constant. Computer simulation of this model produces exactly the same stationary distribution

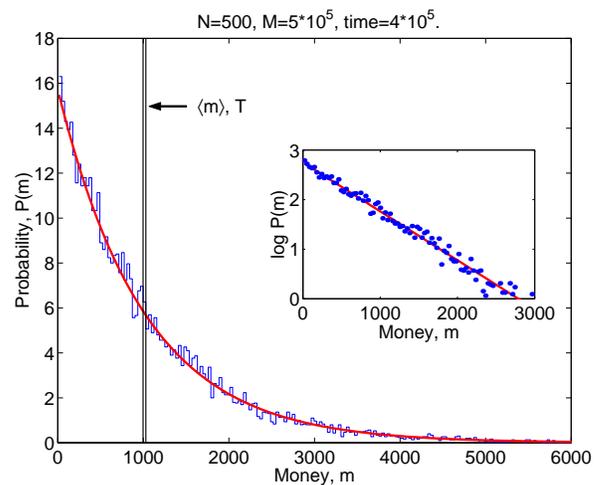


FIG. 1 *Histogram and points*: Stationary probability distribution of money  $P(m)$  obtained in agent-based computer simulations. *Solid curves*: Fits to the Boltzmann-Gibbs law (7). *Vertical line*: The initial distribution of money. From Drăgulescu and Yakovenko (2000).

(7), as in the first model. Computer animation for this model is also given by Chen and Yakovenko (2007).

The final distribution is universal despite different rules for  $\Delta m$ . To amplify this point further, Drăgulescu and Yakovenko (2000) also considered a toy model, where  $\Delta m$  was taken to be a random fraction of the average amount of money of the two agents:  $\Delta m = \nu(m_i + m_j)/2$ . This model produced the same stationary distribution (7) as the two other models.

The models of pairwise money transfer are attractive in their simplicity, but they represent a rather primitive market. Modern economy is dominated by big firms, which consist of many agents, so Drăgulescu and Yakovenko (2000) also studied a model with firms. One agent at a time is appointed to become a “firm”. The firm borrows capital  $K$  from another agent and returns it with interest  $hK$ , hires  $L$  agents and pays them wages  $\omega$ , manufactures  $Q$  items of a product, sells them to  $Q$  agents at a price  $p$ , and receives profit  $F = pQ - \omega L - hK$ . All of these agents are randomly selected. The parameters of the model are optimized following a procedure from economics textbooks (McConnell and Brue, 1996). The aggregate demand-supply curve for the product is taken in the form  $p(Q) = v/Q^\eta$ , where  $Q$  is the quantity consumers would buy at the price  $p$ , and  $\eta$  and  $v$  are some parameters. The production function of the firm has the traditional Cobb-Douglas form:  $Q(L, K) = L^\chi K^{1-\chi}$ , where  $\chi$  is a parameter. Then the profit of the firm  $F$  is maximized with respect to  $K$  and  $L$ . The net result of the firm activity is a many-body transfer of money, which still satisfies the conservation law. Computer simulation of this model generates the same exponential distribution (7), independently of the model parameters. The reasons for the universality of the Boltzmann-Gibbs distribution and

<sup>4</sup> Because debt is not permitted in this model, we have  $M = M_b$ , where  $M_b$  is the monetary base (McConnell and Brue, 1996).

its limitations are discussed in Sec. II.F.

Well after the paper by Drăgulescu and Yakovenko (2000) appeared, the Italian econophysicists Patriarca *et al.* (2005) found that similar ideas had been published earlier in obscure Italian journals by Eleonora Bennati (1988, 1993). It was proposed to call these models the Bennati-Drăgulescu-Yakovenko game (Garibaldi *et al.*, 2007; Scalas *et al.*, 2006). The Boltzmann distribution was independently applied to social sciences by the physicist Jürgen Mimkes (2000); Mimkes and Willis (2005) using the Lagrange principle of maximization with constraints. The exponential distribution of money was also found by the economist Martin Shubik (1999) using a Markov chain approach to strategic market games. A long time ago, Benoit Mandelbrot (1960, p 83) observed:

“There is a great temptation to consider the exchanges of money which occur in economic interaction as analogous to the exchanges of energy which occur in physical shocks between gas molecules.”

He realized that this process should result in the exponential distribution, by analogy with the barometric distribution of density in the atmosphere. However, he discarded this idea, because it does not produce the Pareto power law, and proceeded to study the stable Lévy distributions. Ironically, the actual economic data, discussed in Secs. III.C and IV.A, do show the exponential distribution for the majority of the population. Moreover, the data have a finite variance, so the stable Lévy distributions are not applicable because of their infinite variance.

#### D. Models with debt

Now let us discuss how the results change when debt is permitted.<sup>5</sup> From the standpoint of individual economic agents, debt may be considered as negative money. When an agent borrows money from a bank (considered here as a big reservoir of money),<sup>6</sup> the cash balance of the agent (positive money) increases, but the agent also acquires a debt obligation (negative money), so the total balance (net worth) of the agent remains the same. Thus, the act of borrowing money still satisfies a generalized conservation law of the total money (net worth), which is now defined as the algebraic sum of positive (cash  $M$ ) and

negative (debt  $D$ ) contributions:  $M - D = M_b$ . After spending some cash in binary transactions (5), the agent still has the debt obligation (negative money), so the total money balance  $m_i$  of the agent (net worth) becomes negative. We see that the boundary condition  $m_i \geq 0$ , discussed in Sec. II.B, does not apply when debt is permitted, so  $m = 0$  is not the ground state any more. The consequence of permitting debt is not a violation of the conservation law (which is still preserved in the generalized form for net worth), but a modification of the boundary condition by permitting agents to have negative balances  $m_i < 0$  of net worth. A more detailed discussion of positive and negative money and the book-keeping accounting from the econophysics point of view was presented by the physicist Dieter Braun (2001) and Fischer and Braun (2003a,b).

Now we can repeat the simulation described in Sec. II.C without the boundary condition  $m \geq 0$  by allowing agents to go into debt. When an agent needs to buy a product at a price  $\Delta m$  exceeding his money balance  $m_i$ , the agent is now permitted to borrow the difference from a bank and, thus, to buy the product. As a result of this transaction, the new balance of the agent becomes negative:  $m'_i = m_i - \Delta m < 0$ . Notice that the local conservation law (5) and (6) is still satisfied, but it involves negative values of  $m$ . If the simulation is continued further without any restrictions on the debt of the agents, the probability distribution of money  $P(m)$  never stabilizes, and the system never reaches a stationary state. As time goes on,  $P(m)$  keeps spreading in a Gaussian manner unlimitedly toward  $m = +\infty$  and  $m = -\infty$ . Because of the generalized conservation law discussed above, the first moment  $\langle m \rangle = M_b/N$  of the algebraically defined money  $m$  remains constant. It means that some agents become richer with positive balances  $m > 0$  at the expense of other agents going further into debt with negative balances  $m < 0$ , so that  $M = M_b + D$ .

Common sense, as well as the experience with the current financial crisis, tells us that an economic system cannot be stable if unlimited debt is permitted.<sup>7</sup> In this case, agents can buy any goods without producing anything in exchange by simply going into unlimited debt. Arguably, the current financial crisis was caused by the enormous debt accumulation in the system, triggered by subprime mortgages and financial derivatives based on them. A widely expressed opinion is that the current crisis is not the problem of liquidity, i.e., a temporary difficulty in cash flow, but the problem of insolvency, i.e., the inherent inability of many participants pay back their debts.

Detailed discussion of the current economic situation is not a subject of this paper. Going back to the idealized model of money transfers, one would need to impose some sort of modified boundary conditions in order to prevent

<sup>5</sup> The ideas presented here are quite similar to those by Soddy (1933).

<sup>6</sup> Here we treat the bank as being outside of the system consisting of ordinary agents, because we are interested in money distribution among these agents. The debt of agents is an asset for the bank, and deposits of cash into the bank are liabilities of the bank (McConnell and Brue, 1996). We do not go into these details in order to keep our presentation simple. For more discussion, see Keen (2008).

<sup>7</sup> In qualitative agreement with the conclusions by McCauley (2008).

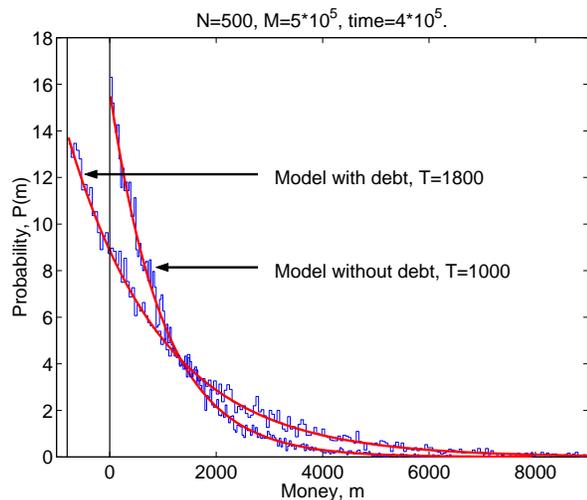


FIG. 2 *Histograms*: Stationary distributions of money with and without debt. The debt is limited to  $m_d = 800$ . *Solid curves*: Fits to the Boltzmann-Gibbs laws with the “money temperatures”  $T_m = 1800$  and  $T_m = 1000$ . From Drăgulescu and Yakovenko (2000).

unlimited growth of debt and to ensure overall stability of the system. Drăgulescu and Yakovenko (2000) considered a simple model where the maximal debt of each agent is limited to a certain amount  $m_d$ . This means that the boundary condition  $m_i \geq 0$  is now replaced by the condition  $m_i \geq -m_d$  for all agents  $i$ . Setting interest rates on borrowed money to be zero for simplicity, Drăgulescu and Yakovenko (2000) performed computer simulations of the models described in Sec. II.C with the new boundary condition. The results are shown in Fig. 2. Not surprisingly, the stationary money distribution again has the exponential shape, but now with the new boundary condition at  $m = -m_d$  and the higher money temperature  $T_d = m_d + M_b/N$ . By allowing agents to go into debt up to  $m_d$ , we effectively increase the amount of money available to each agent by  $m_d$ . So, the money temperature, which is equal to the average amount of effectively available money per agent, increases correspondingly.

Xi, Ding, and Wang (2005) considered another, more realistic boundary condition, where a constraint is imposed not on the individual debt of each agent, but on the total debt of all agents in the system. This is accomplished via the required reserve ratio  $R$ , which is briefly explained below (McConnell and Brue, 1996). Banks are required by law to set aside a fraction  $R$  of the money deposited into bank accounts, whereas the remaining fraction  $1 - R$  can be loaned further. If the initial amount of money in the system (the money base) is  $M_b$ , then, with repeated loans and borrowing, the total amount of positive money available to the agents increases to  $M = M_b/R$ , where the factor  $1/R$  is called the money multiplier (McConnell and Brue, 1996). This is how “banks create money”. Where does this extra

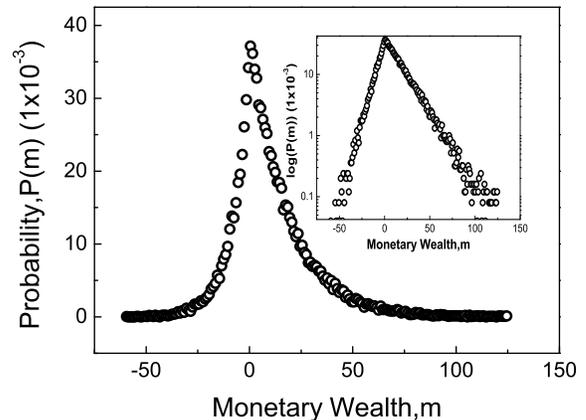


FIG. 3 The stationary distribution of money for the required reserve ratio  $R = 0.8$ . The distribution is exponential for positive and negative money with different “temperatures”  $T_+$  and  $T_-$ , as illustrated by the inset on log-linear scale. From Xi, Ding, and Wang (2005).

money come from? It comes from the increase in the total debt in the system. The maximal total debt is given by  $D = M_b/R - M_b$  and is limited by the factor  $R$ . When the debt is maximal, the total amounts of positive,  $M_b/R$ , and negative,  $M_b(1 - R)/R$ , money circulate among the agents in the system, so there are two constraints in the model considered by Xi, Ding, and Wang (2005). Thus, we expect to see the exponential distributions of positive and negative money characterized by two different temperatures:  $T_+ = M_b/RN$  and  $T_- = M_b(1 - R)/RN$ . This is exactly what was found in computer simulations by Xi, Ding, and Wang (2005), as shown in Fig. 3. Similar two-sided distributions were also found by Fischer and Braun (2003a).

However, in reality, the reserve requirement is not effective in stabilizing total debt in the system, because it applies only to deposits from general public, but not from corporations (O’Brien, 2007).<sup>8</sup> Moreover, there are alternative instruments of debt, including derivatives and various unregulated “financial innovations”. As a result, the total debt is not limited in practice and sometimes can reach catastrophic proportions. Here we briefly discuss several models with non-stationary debt. Thus far, we did not consider the interest rates. Drăgulescu and Yakovenko (2000) studied a simple model with different interest rates for deposits into and loans from a bank. Computer simulations found that money distribution among the agents is still exponential, but the money temperature slowly changes in

<sup>8</sup> Australia does not have reserve requirements, but China actively uses reserve requirements as a tool of monetary policy.

time. Depending on the choice of parameters, the total amount of money in circulation either increases or decreases in time. A more sophisticated macroeconomic model was studied by the economist Steve Keen (1995, 2000). He found that one of the regimes is the debt-induced breakdown, where all economic activity stops under the burden of heavy debt and cannot be restarted without a “debt moratorium”. The interest rates were fixed in these models and not adjusted self-consistently. Cockshott and Cottrell (2008) proposed a mechanism, where the interest rates are set to cover probabilistic withdrawals of deposits from a bank. In an agent-based simulation of the model, Cockshott and Cottrell (2008) found that money supply first increases up to a certain limit, and then the economy experiences a spectacular crash under the weight of accumulated debt. Further studies along these lines would be very interesting. In the rest of the paper, we review various models without debt proposed in literature.

### E. Proportional money transfers and saving propensity

In the models of money transfer discussed in Sec. II.C, the transferred amount  $\Delta m$  is typically independent of the money balances of the agents involved. A different model was introduced in physics literature earlier by Ispolatov, Krapivsky, and Redner (1998) and called the multiplicative asset exchange model. This model also satisfies the conservation law, but the transferred amount of money is a fixed fraction  $\gamma$  of the payer’s money in Eq. (5):

$$\Delta m = \gamma m_i. \quad (8)$$

The stationary distribution of money in this model, compared in Fig. 4 with an exponential function, is similar, but not exactly equal, to the Gamma distribution:

$$P(m) = c m^\beta e^{-m/T}. \quad (9)$$

Eq. (9) differs from Eq. (7) by the power-law prefactor  $m^\beta$ . From the Boltzmann kinetic equation (discussed in Sec. II.F), Ispolatov, Krapivsky, and Redner (1998) derived a formula relating the parameters  $\gamma$  and  $\beta$  in Eqs. (8) and (9):

$$\beta = -1 - \ln 2 / \ln(1 - \gamma). \quad (10)$$

When payers spend a relatively small fraction of their money  $\gamma < 1/2$ , Eq. (10) gives  $\beta > 0$ . In this case, the population with low money balances is reduced, and  $P(0) = 0$ , as shown in Fig. 4.

The economist Thomas Lux (2005) brought to the attention of physicists that essentially the same model, called the inequality process, had been introduced and studied much earlier by the sociologist John Angle (1986, 1992, 1993, 1996, 2002), see also the review by Angle (2006) for additional references. While Ispolatov, Krapivsky, and Redner (1998) did not give

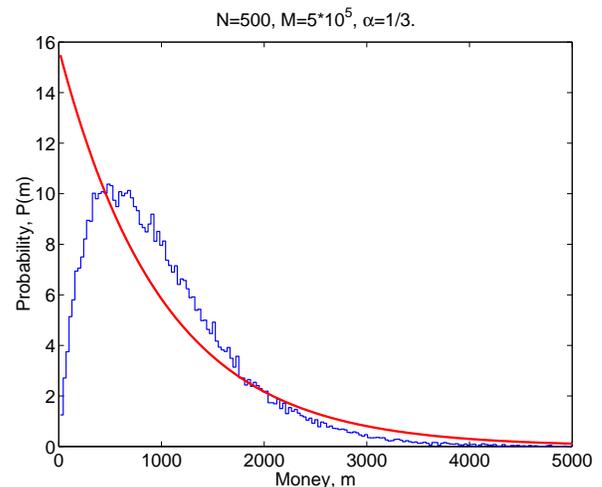


FIG. 4 *Histogram*: Stationary probability distribution of money in the multiplicative random exchange model (8) for  $\gamma = 1/3$ . *Solid curve*: The exponential Boltzmann-Gibbs law. From Drăgulescu and Yakovenko (2000).

much justification for the proportionality law (8), Angle (1986) connected this rule with the surplus theory of social stratification (Engels, 1972), which argues that inequality in human society develops when people can produce more than necessary for minimal subsistence. This additional wealth (surplus) can be transferred from original producers to other people, thus generating inequality. In the first paper by Angle (1986), the parameter  $\gamma$  was randomly distributed, and another parameter  $\delta$  gave a higher probability of winning to the agent with the higher money balance in Eq. (5). However, in the following papers, he simplified the model to a fixed  $\gamma$  (denoted as  $\omega$  by Angle) and equal probabilities of winning for higher- and lower-balance agents, which makes it completely equivalent to the model of Ispolatov, Krapivsky, and Redner (1998). Angle (2002, 2006) also considered a model where groups of agents have different values of  $\gamma$ , simulating the effect of education and other “human capital”. All of these models generate a Gamma-like distribution, well approximated by Eq. (9).

Another model with an element of proportionality was proposed by Chakraborti and Chakrabarti (2000).<sup>9</sup> In this model, the agents set aside (save) some fraction of their money  $\lambda m_i$ , whereas the rest of their money balance  $(1 - \lambda)m_i$  becomes available for random exchanges. Thus, the rule of exchange (5) becomes

$$\begin{aligned} m'_i &= \lambda m_i + \xi(1 - \lambda)(m_i + m_j), \\ m'_j &= \lambda m_j + (1 - \xi)(1 - \lambda)(m_i + m_j). \end{aligned} \quad (11)$$

Here the coefficient  $\lambda$  is called the saving propensity, and

<sup>9</sup> This paper originally appeared as a follow-up e-print cond-mat/0004256 on the e-print cond-mat/0001432 by Drăgulescu and Yakovenko (2000).

the random variable  $\xi$  is uniformly distributed between 0 and 1. It was pointed out by Angle (2006) that, by the change of notation  $\lambda \rightarrow (1 - \gamma)$ , Eq. (11) can be transformed to the same form as Eq. (8), if the random variable  $\xi$  takes only discrete values 0 and 1. Computer simulations by Chakraborti and Chakrabarti (2000) of the model (11) found a stationary distribution close to the Gamma distribution (9). It was shown that the parameter  $\beta$  is related to the saving propensity  $\lambda$  by the formula  $\beta = 3\lambda/(1-\lambda)$  (Patriarca, Chakraborti, and Kaski, 2004a,b; Patriarca *et al.*, 2005; Repetowicz, Hutzler, and Richmond, 2005). For  $\lambda \neq 0$ , agents always keep some money, so their balances never drop to zero, and  $P(0) = 0$ , whereas for  $\lambda = 0$  the distribution becomes exponential.

In the subsequent papers by the Kolkata school (Chakrabarti, 2005) and related papers, the case of random saving propensity was studied. In these models, the agents are assigned random parameters  $\lambda$  drawn from a uniform distribution between 0 and 1 (Chatterjee, Chakrabarti, and Manna, 2004). It was found that this model produces a power-law tail  $P(m) \propto 1/m^2$  at high  $m$ . The reasons for stability of this law were understood using the Boltzmann kinetic equation (Chatterjee, Chakrabarti, and Stinchcombe, 2005; Das and Yarlagadda, 2005; Repetowicz, Hutzler, and Richmond, 2005), but most elegantly in the mean-field theory (Bhattacharyya, Chatterjee, and Chakrabarti, 2007; Chatterjee and Chakrabarti, 2007; Mohanty, 2006). The fat tail originates from the agents whose saving propensity is close to 1, who hoard money and do not give it back (Patriarca, Chakraborti, and Germano, 2006; Patriarca *et al.*, 2005). A more rigorous mathematical treatment of the problem was given by Düring, Matthes, and Toscani (2008); Düring and Toscani (2007); Matthes and Toscani (2008). An interesting matrix formulation of the problem was presented by Gupta (2006). Relaxation rate in the money transfer models was studied by Düring, Matthes, and Toscani (2008); Gupta (2008); Patriarca *et al.* (2007). Drăgulescu and Yakovenko (2000) considered a model with taxation, which also has an element of proportionality. The Gamma distribution was also studied for conservative models within a simple Boltzmann approach by Ferrero (2004) and, using more complicated rules of exchange motivated by political economy, by Scafetta, Picozzi, and West (2004a,b). Independently, the economist Miguel Molico (2006) studied conservative exchange models where agents bargain over prices in their transactions. He found stationary Gamma-like distributions of money in numerical simulations of these models.

## F. Additive versus multiplicative models

The stationary distribution of money (9) for the models of Sec. II.E is different from the simple exponential formula (7) found for the models of Sec. II.C. The origin of this difference can be understood from the Boltzmann kinetic equation (Lifshitz and Pitaevskii, 1981; Wannier, 1987). This equation describes time evolution of the distribution function  $P(m)$  due to pairwise interactions:

$$\frac{dP(m)}{dt} = \iint \{-f_{[m,m'] \rightarrow [m-\Delta, m'+\Delta]} P(m) P(m') + f_{[m-\Delta, m'+\Delta] \rightarrow [m, m']} P(m-\Delta) P(m'+\Delta)\} dm' d\Delta. \quad (12)$$

Here  $f_{[m,m'] \rightarrow [m-\Delta, m'+\Delta]}$  is the probability of transferring money  $\Delta$  from an agent with money  $m$  to an agent with money  $m'$  per unit time. This probability, multiplied by the occupation numbers  $P(m)$  and  $P(m')$ , gives the rate of transitions from the state  $[m, m']$  to the state  $[m-\Delta, m'+\Delta]$ . The first term in Eq. (12) gives the depopulation rate of the state  $m$ . The second term in Eq. (12) describes the reversed process, where the occupation number  $P(m)$  increases. When the two terms are equal, the direct and reversed transitions cancel each other statistically, and the probability distribution is stationary:  $dP(m)/dt = 0$ . This is the principle of detailed balance.

In physics, the fundamental microscopic equations of motion obey the time-reversal symmetry. This means that the probabilities of the direct and reversed processes are exactly equal:

$$f_{[m,m'] \rightarrow [m-\Delta, m'+\Delta]} = f_{[m-\Delta, m'+\Delta] \rightarrow [m, m']}. \quad (13)$$

When Eq. (13) is satisfied, the detailed balance condition for Eq. (12) reduces to the equation  $P(m)P(m') = P(m-\Delta)P(m'+\Delta)$ , because the factors  $f$  cancels out. The only solution of this equation is the exponential function  $P(m) = c \exp(-m/T_m)$ , so the Boltzmann-Gibbs distribution is the stationary solution of the Boltzmann kinetic equation (12). Notice that the transition probabilities (13) are determined by the dynamical rules of the model, but the equilibrium Boltzmann-Gibbs distribution does not depend on the dynamical rules at all. This is the origin of the universality of the Boltzmann-Gibbs distribution. We see that it is possible to find the stationary distribution without knowing details of the dynamical rules (which are rarely known very well), as long as the symmetry condition (13) is satisfied.

The models considered in Sec. II.C have the time-reversal symmetry. The model with the fixed money transfer  $\Delta$  has equal probabilities (13) of transferring money from an agent with the balance  $m$  to an agent with the balance  $m'$  and vice versa. This is also true when  $\Delta$  is random, as long as the probability distribution of  $\Delta$  is independent of  $m$  and  $m'$ . Thus, the stationary distribution  $P(m)$  is always exponential in these models.

However, there is no fundamental reason to expect the time-reversal symmetry in economics, where Eq. (13) may be not valid. In this case, the system may have a

non-exponential stationary distribution or no stationary distribution at all. In the model (8), the time-reversal symmetry is broken. Indeed, when an agent  $i$  gives a fixed fraction  $\gamma$  of his money  $m_i$  to an agent with balance  $m_j$ , their balances become  $(1-\gamma)m_i$  and  $m_j + \gamma m_i$ . If we try to reverse this process and appoint the agent  $j$  to be the payer and to give the fraction  $\gamma$  of her money,  $\gamma(m_j + \gamma m_i)$ , to the agent  $i$ , the system does not return to the original configuration  $[m_i, m_j]$ . As emphasized by Angle (2006), the payer pays a deterministic fraction of his money, but the receiver receives a random amount from a random agent, so their roles are not interchangeable. Because the proportional rule typically violates the time-reversal symmetry, the stationary distribution  $P(m)$  in multiplicative models is typically not exponential.<sup>10</sup> Making the transfer dependent on the money balance of the payer effectively introduces Maxwell's demon into the model. Another view on the time-reversal symmetry in economic dynamics was presented by Ao (2007).

These examples show that the Boltzmann-Gibbs distribution does not necessarily hold for any conservative model. However, it is universal in a limited sense. For a broad class of models that have time-reversal symmetry, the stationary distribution is exponential and does not depend on details of a model. Conversely, when the time-reversal symmetry is broken, the distribution may depend on details of a model. The difference between these two classes of models may be rather subtle. Deviations from the Boltzmann-Gibbs law may occur only if the transition rates  $f$  in Eq. (13) explicitly depend on the agents' money  $m$  or  $m'$  in an asymmetric manner. Drăgulescu and Yakovenko (2000) performed a computer simulation where the direction of payment was randomly fixed in advance for every pair of agents  $(i, j)$ . In this case, money flows along directed links between the agents:  $i \rightarrow j \rightarrow k$ , and the time-reversal symmetry is strongly violated. This model is closer to the real economy, where one typically receives money from an employer and pays it to a grocery store. Nevertheless, the Boltzmann-Gibbs distribution was still found in this model, because the transition rates  $f$  do not explicitly depend on  $m$  and  $m'$  and do not violate Eq. (13). A more general study of money exchange models on directed networks was presented by Chatterjee (2009).

In the absence of detailed knowledge of real microscopic dynamics of economic exchanges, the semi-universal Boltzmann-Gibbs distribution (7) is a natural starting point. Moreover, the assumption of Drăgulescu and Yakovenko (2000) that agents pay the same prices  $\Delta m$  for the same products, independent of their money balances  $m$ , seems very appropriate for the modern anonymous economy, especially for purchases

over the Internet. There is no particular empirical evidence for the proportional rules (8) or (11). However, the difference between the additive (7) and multiplicative (9) distributions may be not so crucial after all. From the mathematical point of view, the difference is in the implementation of the boundary condition at  $m = 0$ . In the additive models of Sec. II.C, there is a sharp cutoff for  $P(m) \neq 0$  at  $m = 0$ . In the multiplicative models of Sec. II.E, the balance of an agent never reaches  $m = 0$ , so  $P(m)$  vanishes at  $m \rightarrow 0$  in a power-law manner. But for large  $m$ ,  $P(m)$  decreases exponentially in both models.

By further modifying the rules of money transfer and introducing more parameters in the models, one can obtain even more complicated distributions (Saif and Gade, 2007; Scafetta and West, 2007). However, one can argue that parsimony is the virtue of a good mathematical model, not the abundance of additional assumptions and parameters, whose correspondence to reality is hard to verify.

### III. STATISTICAL MECHANICS OF WEALTH DISTRIBUTION

In the econophysics literature on exchange models, the terms “money” and “wealth” are often used interchangeably. However, economists emphasize the difference between these two concepts. In this section, we review the models of wealth distribution, as opposed to money distribution.

#### A. Models with a conserved commodity

What is the difference between money and wealth? Drăgulescu and Yakovenko (2000) argued that wealth  $w_i$  is equal to money  $m_i$  plus the other property that an agent  $i$  has. The latter may include durable material property, such as houses and cars, and financial instruments, such as stocks, bonds, and options. Money (paper cash, bank accounts) is generally liquid and countable. However, the other property is not immediately liquid and has to be sold first (converted into money) to be used for other purchases. In order to estimate the monetary value of property, one needs to know its price  $p$ . In the simplest model, let us consider just one type of property, say, stocks  $s$ . Then the wealth of an agent  $i$  is given by

$$w_i = m_i + p s_i. \quad (14)$$

It is assumed that the price  $p$  is common for all agents and is established by some kind of market process, such as an auction, and may change in time.

It is reasonable to start with a model where both the total money  $M = \sum_i m_i$  and the total stock  $S = \sum_i s_i$  are conserved (Ausloos and Pekalski, 2007; Chakraborti, Pradhan, and Chakraborti, 2001; Chatterjee and Chakraborti, 2006). The agents pay

<sup>10</sup> However, when  $\Delta m$  is a fraction of the total money  $m_i + m_j$  of the two agents, the model is time-reversible and has the exponential distribution, as discussed in Sec. II.C.

money to buy stock and sell stock to get money, and so on. Although  $M$  and  $S$  are conserved, the total wealth  $W = \sum_i w_i$  is generally not conserved (Chatterjee and Chakrabarti, 2006), because of price fluctuation in Eq. (14). This is an important difference from the money transfers models of Sec. II. The wealth  $w_i$  of an agent  $i$ , not participating in any transactions, may change when transactions between other agents establish a new price  $p$ . Moreover, the wealth  $w_i$  of an agent  $i$  does not change after a transaction with an agent  $j$ . Indeed, in exchange for paying money  $\Delta m$ , the agent  $i$  receives the stock  $\Delta s = \Delta m/p$ , so her total wealth (14) remains the same. Theoretically, the agent can instantaneously sell the stock back at the same price and recover the money paid. If the price  $p$  never changes, then the wealth  $w_i$  of each agent remains constant, despite transfers of money and stock between agents.

We see that redistribution of wealth in this model is directly related to price fluctuations. A mathematical model of this process was studied by Silver, Slud, and Takamoto (2002). In this model, the agents randomly change preferences for the fraction of their wealth invested in stocks. As a result, some agents offer stock for sale and some want to buy it. The price  $p$  is determined from the market-clearing auction matching supply and demand. Silver, Slud, and Takamoto (2002) demonstrated in computer simulations and proved analytically using the theory of Markov processes that the stationary distribution  $P(w)$  of wealth  $w$  in this model is given by the Gamma distribution, as in Eq. (9). Various modifications of this model considered by Lux (2005), such as introducing monopolistic coalitions, do not change this result significantly, which shows robustness of the Gamma distribution. For models with a conserved commodity, Chatterjee and Chakrabarti (2006) found the Gamma distribution for a fixed saving propensity and a power-law tail for a distributed saving propensity.

Another model with conserved money and stock was studied by Raberto *et al.* (2003) for an artificial stock market, where traders follow different investment strategies: random, momentum, contrarian, and fundamentalist. Wealth distribution in the model with random traders was found have a power-law tail  $P(w) \sim 1/w^2$  for large  $w$ . However, unlike in other simulations, where all agents initially have equal balances, here the initial money and stock balances of the agents were randomly populated according to a power law with the same exponent. This raises the question whether the observed power-law distribution of wealth is an artifact of the initial conditions, because equilibration of the upper tail may take a very long simulation time.

## B. Models with stochastic growth of wealth

Although the total wealth  $W$  is not exactly conserved in the models considered in Sec. III.A, never-

theless  $W$  remains constant on average, because the total money  $M$  and stock  $S$  are conserved. A different model for wealth distribution was proposed by Bouchaud and Mézard (2000). In this model, time evolution of the wealth  $w_i$  of an agent  $i$  is given by the stochastic differential equation

$$\frac{dw_i}{dt} = \eta_i(t) w_i + \sum_{j(\neq i)} J_{ij} w_j - \sum_{j(\neq i)} J_{ji} w_i, \quad (15)$$

where  $\eta_i(t)$  is a Gaussian random variable with the mean  $\langle \eta \rangle$  and the variance  $2\sigma^2$ . This variable represents growth or loss of wealth of an agent due to investment in stock market. The last two terms describe transfer of wealth between different agents, which is taken to be proportional to the wealth of the payers with the coefficients  $J_{ij}$ . So, the model (15) is multiplicative and invariant under the scale transformation  $w_i \rightarrow Z w_i$ . For simplicity, the exchange fractions are taken to be the same for all agents:  $J_{ij} = J/N$  for all  $i \neq j$ , where  $N$  is the total number of agents. In this case, the last two terms in Eq. (15) can be written as  $J(\langle w \rangle - w_i)$ , where  $\langle w \rangle = \sum_i w_i/N$  is the average wealth per agent. This case represents a “mean-field” model, where all agents feel the same environment. It can be easily shown that the average wealth increases in time as  $\langle w \rangle_t = \langle w \rangle_0 e^{(\langle \eta \rangle + \sigma^2)t}$ . Then, it makes more sense to consider the relative wealth  $\tilde{w}_i = w_i/\langle w \rangle_t$ . Eq. (15) for this variable becomes

$$\frac{d\tilde{w}_i}{dt} = (\eta_i(t) - \langle \eta \rangle - \sigma^2) \tilde{w}_i + J(1 - \tilde{w}_i). \quad (16)$$

The probability distribution  $P(\tilde{w}, t)$  for the stochastic differential equation (16) is governed by the Fokker-Planck equation

$$\frac{\partial P}{\partial t} = \frac{\partial [J(\tilde{w} - 1) + \sigma^2 \tilde{w}] P}{\partial \tilde{w}} + \sigma^2 \frac{\partial}{\partial \tilde{w}} \left( \tilde{w} \frac{\partial (\tilde{w} P)}{\partial \tilde{w}} \right). \quad (17)$$

The stationary solution ( $\partial P/\partial t = 0$ ) of this equation is given by the following formula

$$P(\tilde{w}) = c \frac{e^{-J/\sigma^2 \tilde{w}}}{\tilde{w}^2 + J/\sigma^2}. \quad (18)$$

The distribution (18) is quite different from the Boltzmann-Gibbs (7) and Gamma (9) distributions. Eq. (18) has a power-law tail at large  $\tilde{w}$  and a sharp cutoff at small  $\tilde{w}$ . Eq. (15) is a version of the generalized Lotka-Volterra model, and the stationary distribution (18) was also obtained by Solomon and Richmond (2001, 2002). The model was generalized to include negative wealth by Huang (2004).

Bouchaud and Mézard (2000) used the mean-field approach. A similar result was found for a model with pairwise interaction between agents by Slanina (2004). In his model, wealth is transferred between the agents following the proportional rule (8), but, in addition, the wealth of the agents increases by the factor  $1 + \zeta$  in each

transaction. This factor is supposed to reflect creation of wealth in economic interactions. Because the total wealth in the system increases, it makes sense to consider the distribution of relative wealth  $P(\tilde{w})$ . In the limit of continuous trading, Slanina (2004) found the same stationary distribution (18). This result was reproduced using a mathematically more involved treatment of this model by Cordier, Pareschi, and Toscani (2005); Pareschi and Toscani (2006). Numerical simulations of the models with stochastic noise  $\eta$  by Scafetta, Picozzi, and West (2004a,b) also found a power law tail for large  $w$ . Equivalence between the models with pairwise exchange and exchange with a reservoir was discussed by Basu and Mohanty (2008).

We now contrast the models discussed in Secs. III.A and III.B. In the former case, where money and commodity are conserved, and wealth does not grow, the distribution of wealth is given by the Gamma distribution with the exponential tail for large  $w$ . In the latter models, wealth grows in time exponentially, and the distribution of relative wealth has a power-law tail for large  $\tilde{w}$ . These results suggest that the presence of a power-law tail is a nonequilibrium effect that requires constant growth or inflation of the economy, but disappears for a closed system with conservation laws.

The discussed models were reviewed by Chatterjee and Chakrabarti (2007); Richmond, Hutzler, Coelho, and Repetowicz (2006); Richmond, Repetowicz, Hutzler, and Coelho (2006); Yakovenko (2009) and in the popular article by Hayes (2002). Because of lack of space, we omit discussion of models with wealth condensation (Bouchaud and Mézard, 2000; Braun, 2006; Burda et al., 2002; Ispolatov, Krapivsky, and Redner, 1998; Pianegonda *et al.*, 2003), where a few agents accumulate a finite fraction of the total wealth, and studies of wealth distribution on complex networks (Coelho *et al.*, 2005; Di Matteo, Aste, and Hyde, 2004; Hu *et al.*, 2006, 2007; Iglesias *et al.*, 2003). So far, we discussed the models with long-range interaction, where any agent can exchange money and wealth with any other agent. A local model, where agents trade only with the nearest neighbors, was studied by Bak, Nørrelykke, and Shubik (1999).

### C. Empirical data on money and wealth distributions

It would be interesting to compare theoretical results for money and wealth distributions in various models with empirical data. Unfortunately, such empirical data are difficult to find. Unlike income, which is discussed in Sec. IV, wealth is not routinely reported by the majority of individuals to the government. However, in some countries, when a person dies, all assets must be reported for the purpose of inheritance tax. So, in principle, there exist good statistics of wealth distribution among dead people, which, of course, is different from the wealth distribution

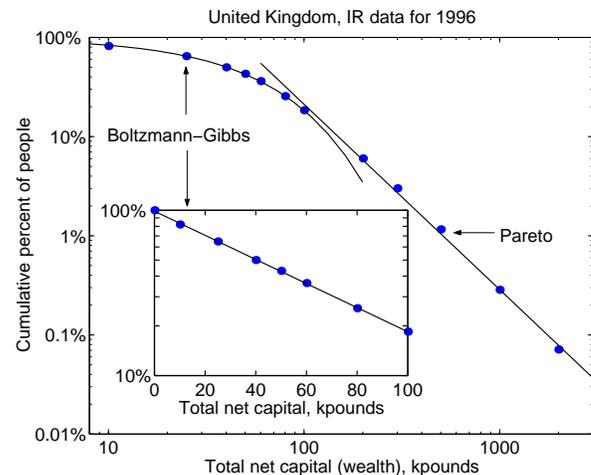


FIG. 5 Cumulative probability distribution of net wealth in the UK shown on log-log (main panel) and log-linear (inset) scales. Points represent the data from the Inland Revenue, and solid lines are fits to the exponential (Boltzmann-Gibbs) and power (Pareto) laws. From Drăgulescu and Yakovenko (2001b).

among the living. Using an adjustment procedure based on the age, gender, and other characteristics of the deceased, the UK tax agency, the Inland Revenue, reconstructed the wealth distribution of the whole population of the UK (Her Majesty Revenue and Customs, 2003). Fig. 5 shows the UK data for 1996 reproduced from Drăgulescu and Yakovenko (2001b). The figure shows the cumulative probability  $C(w) = \int_w^\infty P(w') dw'$  as a function of the personal net wealth  $w$ , which is composed of assets (cash, stocks, property, household goods, etc.) and liabilities (mortgages and other debts). Because statistical data are usually reported at non-uniform intervals of  $w$ , it is more practical to plot the cumulative probability distribution  $C(w)$  rather than its derivative, the probability density  $P(w)$ . Fortunately, when  $P(w)$  is an exponential or a power-law function, then  $C(w)$  is also an exponential or a power-law function.

The main panel in Fig. 5 shows a plot of  $C(w)$  on the log-log scale, where a straight line represents a power-law dependence. The figure shows that the distribution follows a power law  $C(w) \propto 1/w^\alpha$  with the exponent  $\alpha = 1.9$  for the wealth greater than about 100 k $\mathcal{L}$ . The inset in Fig. 5 shows the same data on the log-linear scale, where a straight line represents an exponential dependence. We observe that, below 100 k $\mathcal{L}$ , the data are well fitted by the exponential distribution  $C(w) \propto \exp(-w/T_w)$  with the effective “wealth temperature”  $T_w = 60$  k $\mathcal{L}$  (which corresponds to the median wealth of 41 k $\mathcal{L}$ ). So, the distribution of wealth is characterized by the Pareto power law in the upper tail of the distribution and the exponential Boltzmann-Gibbs law in the lower part of the distribution for the great majority (about 90%) of the population. Similar results are found for the distribution of income, as discussed in

Sec. IV. One may speculate that wealth distribution in the lower part is dominated by distribution of money, because the corresponding people do not have other significant assets (Levy and Levy, 2003), so the results of Sec. II give the Boltzmann-Gibbs law. On the other hand, the upper tail of wealth distribution is dominated by investment assets (Levy and Levy, 2003), where the results of Sec. III.B give the Pareto law. The power law was studied by many researchers (Klass *et al.*, 2007; Levy, 2003; Levy and Levy, 2003; Sinha, 2006) for the upper-tail data, such as the Forbes list of 400 richest people. On the other hand, statistical surveys of the population, such as the Survey of Consumer Finance (Diaz-Giménez *et al.*, 1997) and the Panel Study of Income Dynamics (PSID), give more information about the lower part of the wealth distribution. Curiously, Abul-Magd (2002) found that the wealth distribution in the ancient Egypt was consistent with Eq. (18). Hegyi *et al.* (2007) found a power-law tail for the wealth distribution of aristocratic families in medieval Hungary.

For direct comparison with the results of Sec. II, it would be interesting to find data on the distribution of money, as opposed to the distribution of wealth. Making a reasonable assumption that most people keep most of their money in banks, one can approximate the distribution of money by the distribution of balances on bank accounts. (Balances on all types of bank accounts, such as checking, saving, and money manager, associated with the same person should be added up.) Despite imperfections (people may have accounts in different banks or not keep all their money in banks), the distribution of balances on bank accounts would give valuable information about the distribution of money. The data for a large enough bank would be representative of the distribution in the whole economy. Unfortunately, it has not been possible to obtain such data thus far, even though it would be completely anonymous and not compromise privacy of bank clients.

The data on the distribution of bank accounts balances would be useful, e.g., to the Federal Deposits Insurance Company (FDIC) of the USA. This government agency insures bank deposits of customers up to a certain maximal balance. In order to estimate its exposure and the change in exposure due to a possible increase in the limit, FDIC would need to know the probability distribution of balances on bank accounts. It is quite possible that FDIC may already have such data.

Measuring the probability distribution of money would be also very useful for determining how much people can, in principle, spend on purchases (without going into debt). This is different from the distribution of wealth, where the property component, such as a house, a car, or retirement investment, is effectively locked up and, in most cases, is not easily available for consumer spending. Thus, although wealth distribution may reflect the distribution of economic power, the distribution of money is more relevant for immediate consumption.

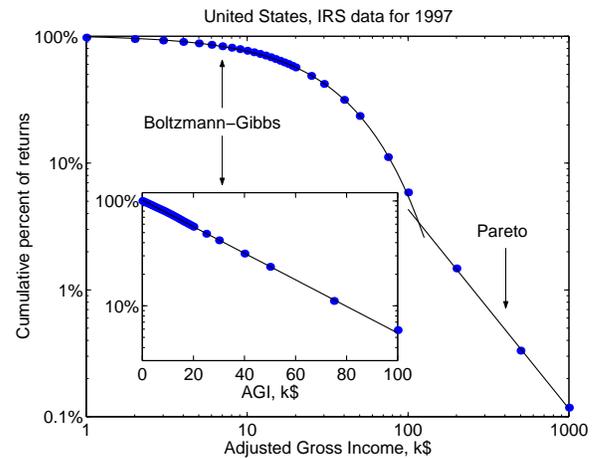


FIG. 6 Cumulative probability distribution of tax returns for USA in 1997 shown on log-log (main panel) and log-linear (inset) scales. Points represent the Internal Revenue Service data, and solid lines are fits to the exponential and power-law functions. From Drăgulescu and Yakovenko (2003).

#### IV. DATA AND MODELS FOR INCOME DISTRIBUTION

In contrast to money and wealth distributions, more empirical data are available for the distribution of income  $r$  from tax agencies and population surveys. In this section, we first present empirical data on income distribution and then discuss theoretical models.

##### A. Empirical data on income distribution

Empirical studies of income distribution have a long history in the economic literature.<sup>11</sup> Many articles on this subject appear in the journal *Review of Income and Wealth*, published on behalf of the International Association for Research in Income and Wealth. Following the work by Pareto (1897), much attention was focused on the power-law upper tail of income distribution and less on the lower part. In contrast to more complicated functions discussed in the economic literature (Atkinson and Bourguignon, 2000; Champernowne and Cowell, 1998; Kakwani, 1980), Drăgulescu and Yakovenko (2001a) demonstrated that the lower part of income distribution can be well fitted with the simple exponential function  $P(r) = c \exp(-r/T_r)$ , which is characterized by just one parameter, the “income temperature”  $T_r$ . Then Drăgulescu and Yakovenko (2001b, 2003) showed that the whole income distribution can be fitted by an exponential function in the lower part and a power-law func-

<sup>11</sup> See, e.g., Atkinson and Bourguignon (2000); Atkinson and Piketty (2007); Champernowne and Cowell (1998); Kakwani (1980); Piketty and Saez (2003).

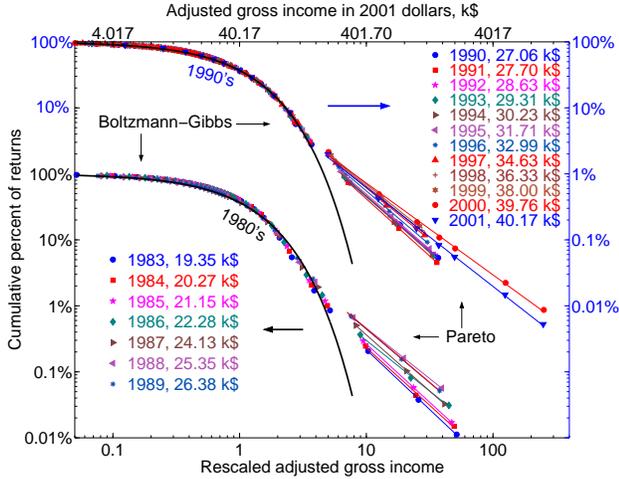


FIG. 7 Cumulative probability distribution of tax returns plotted on log-log scale versus  $r/T_r$  (the annual income  $r$  normalized by the average income  $T_r$  in the exponential part of the distribution). The IRS data points are for 1983–2001, and the columns of numbers give the values of  $T_r$  for the corresponding years. From Silva and Yakovenko (2005).

tion in the upper part, as shown in Fig. 6. The straight line on the log-linear scale in the inset of Fig. 6 demonstrates the exponential Boltzmann-Gibbs law, and the straight line on the log-log scale in the main panel illustrates the Pareto power law. The fact that income distribution consists of two distinct parts reveals the two-class structure of the American society (Silva and Yakovenko, 2005; Yakovenko and Silva, 2005). Coexistence of the exponential and power-law distributions is also known in plasma physics and astrophysics, where they are called the “thermal” and “superthermal” parts (Collier, 2004; Desai *et al.*, 2003; Hasegawa *et al.*, 1985). The boundary between the lower and upper classes can be defined as the intersection point of the exponential and power-law fits in Fig. 6. For 1997, the annual income separating the two classes was about 120 k\$. About 3% of the population belonged to the upper class, and 97% belonged to the lower class.

Silva and Yakovenko (2005) studied time evolution of income distribution in the USA during 1983–2001 using the data from the Internal Revenue Service (IRS), the government tax agency. The structure of income distribution was found to be qualitatively the same for all years, as shown in Fig. 7. The average income in nominal dollars has approximately doubled during this time interval. So, the horizontal axis in Fig. 7 shows the normalized income  $r/T_r$ , where the “income temperature”  $T_r$  was obtained by fitting of the exponential part of the distribution for each year. The values of  $T_r$  are shown in Fig. 7. The plots for the 1980s and 1990s are shifted vertically for clarity. We observe that the data points in the lower-income part of the distribution collapse on the same exponential curve for all years. This demonstrates that the shape of the income distribution for the lower

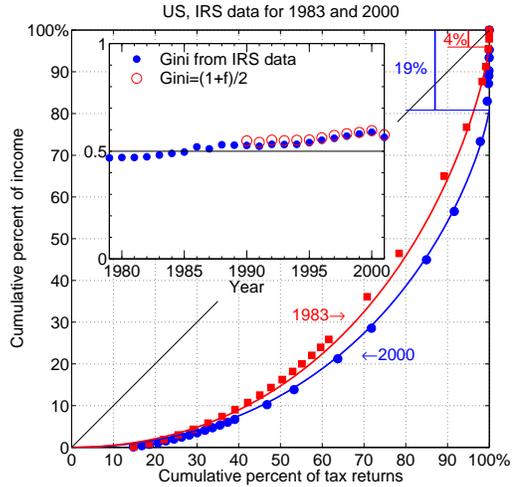


FIG. 8 *Main panel:* Lorenz plots for income distribution in 1983 and 2000. The data points are from the IRS (Strudler, Petska, and Petska, 2003), and the theoretical curves represent Eq. (20) with the parameter  $f$  deduced from Fig. 7. *Inset:* The closed circles are the IRS data (Strudler, Petska, and Petska, 2003) for the Gini coefficient  $G$ , and the open circles show the theoretical formula  $G = (1 + f)/2$ . From Silva and Yakovenko (2005).

class is extremely stable and does not change in time, despite gradual increase in the average income in nominal dollars. This observation suggests that the lower-class distribution is in statistical “thermal” equilibrium.

On the other hand, as Fig. 7 shows, income distribution of the upper class does not rescale and significantly changes in time. Silva and Yakovenko (2005) found that the exponent  $\alpha$  of the power law  $C(r) \propto 1/r^\alpha$  decreased from 1.8 in 1983 to 1.4 in 2000. This means that the upper tail became “fatter”. Another useful parameter is the total income of the upper class as the fraction  $f$  of the total income in the system. The fraction  $f$  increased from 4% in 1983 to 20% in 2000 (Silva and Yakovenko, 2005). However, in year 2001,  $\alpha$  increased and  $f$  decreased, indicating that the upper tail was reduced after the stock market crash at that time. These results indicate that the upper tail is highly dynamical and not stationary. It tends to swell during the stock market boom and shrink during the bust. Similar results were found for Japan (Aoyama *et al.*, 2003; Fujiwara *et al.*, 2003; Souma, 2001, 2002).

Although relative income inequality within the lower class remains stable, the overall income inequality in the USA has increased significantly as a result of the tremendous growth of the income of the upper class. This is illustrated by the Lorenz curve and the Gini coefficient shown in Fig. 8. The Lorenz curve (Kakwani, 1980) is a standard way of representing income distribution in the economic literature. It is defined in terms of two coordi-

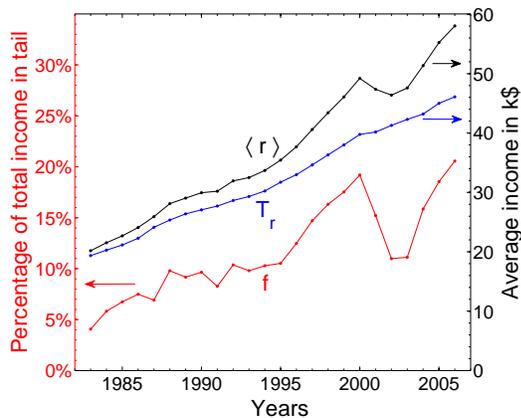


FIG. 9 Historical evolution of the parameters  $\langle r \rangle$ ,  $T_r$ , and the fraction of income  $f$  going to the upper tail, as defined in Eq. (21). From Banerjee (2008).

nates  $x(r)$  and  $y(r)$  depending on a parameter  $r$ :

$$x(r) = \int_0^r P(r') dr', \quad y(r) = \frac{\int_0^r r' P(r') dr'}{\int_0^\infty r' P(r') dr'}. \quad (19)$$

The horizontal coordinate  $x(r)$  is the fraction of the population with income below  $r$ , and the vertical coordinate  $y(r)$  is the fraction of the income this population accounts for. As  $r$  changes from 0 to  $\infty$ ,  $x$  and  $y$  change from 0 to 1 and parametrically define a curve in the  $(x, y)$  plane.

Fig. 8 shows the data points for the Lorenz curves in 1983 and 2000, as computed by the IRS (Strudler, Petska, and Petska, 2003). Drăgulescu and Yakovenko (2001a) analytically derived the Lorenz curve formula  $y = x + (1 - x) \ln(1 - x)$  for a purely exponential distribution  $P(r) = c \exp(-r/T_r)$ . This formula is shown by the upper curve in Fig. 8 and describes the 1983 data reasonably well. However, for year 2000, it is essential to take into account the fraction  $f$  of income in the upper tail, which modifies for the Lorenz formula as follows (Drăgulescu and Yakovenko, 2003; Silva and Yakovenko, 2005; Yakovenko and Silva, 2005)

$$y = (1 - f)[x + (1 - x) \ln(1 - x)] + f \Theta(x - 1). \quad (20)$$

The last term in Eq. (20) represent the vertical jump of the Lorenz curve at  $x = 1$ , where a small percentage of population in the upper class accounts for a substantial fraction  $f$  of the total income. The lower curve in Fig. 8 shows that Eq. (20) fits the 2000 data very well.

The deviation of the Lorenz curve from the straight diagonal line in Fig. 8 is a certain measure of income inequality. Indeed, if everybody had the same income, the Lorenz curve would be the diagonal line, because the fraction of income would be proportional to the fraction of the population. The standard measure of income inequality is the Gini coefficient  $0 \leq G \leq 1$ , which is defined as the area between the Lorenz curve and the

diagonal line, divided by the area of the triangle beneath the diagonal line (Kakwani, 1980). Time evolution of the Gini coefficient, as computed by the IRS (Strudler, Petska, and Petska, 2003), is shown in the inset of Fig. 8. Drăgulescu and Yakovenko (2001a) derived analytically the result that  $G = 1/2$  for a purely exponential distribution. In the first approximation, the values of  $G$  shown in the inset of Fig. 8 are indeed close to the theoretical value  $1/2$ . If we take into account the upper tail using Eq. (20), the formula for the Gini coefficient becomes  $G = (1 + f)/2$  (Silva and Yakovenko, 2005). The inset in Fig. 8 shows that this formula gives a very good fit to the IRS data for the 1990s using the values of  $f$  deduced from Fig. 7. The values  $G < 1/2$  in the 1980s cannot be captured by this formula, because the Lorenz data points are slightly above the theoretical curve for 1983 in Fig. 8. Overall, we observe that income inequality has been increasing for the last 20 years, because of swelling of the Pareto tail, but decreased in 2001 after the stock market crash.

It is easy to show that the parameter  $f$  in Eq. (20) and in Fig. 8 is given by

$$f = \frac{\langle r \rangle - T_r}{\langle r \rangle}, \quad (21)$$

where  $\langle r \rangle$  is the average income of the whole population, and the temperature  $T_r$  is the average income in the exponential part of the distribution. Eq. (21) gives a well-defined measure of the deviation of the actual income distribution from the exponential one and, thus, of the fatness of the upper tail. Fig. 9 shows historical evolution of the parameters  $\langle r \rangle$ ,  $T_r$ , and  $f$  given by Eq. (21).<sup>12</sup> We observe that  $T_r$  has been increasing, essentially, monotonously (most of this increase is inflation). In contrast,  $\langle r \rangle$  had sharp peaks in 2000 and 2006 coinciding with the speculative bubbles in financial markets. The fraction  $f$ , which characterizes income inequality, has been increasing for the last 20 years and reached maxima of 20% in the years 2000 and 2006 with a sharp drop in between. We conclude that the speculative bubbles greatly increase the fraction of income going to the upper tail, but do not change income distribution of the lower class. When the bubbles inevitably collapse, income inequality reduces.

Thus far we discussed the distribution of individual income. An interesting related question is the distribution  $P_2(r)$  of family income  $r = r_1 + r_2$ , where  $r_1$  and  $r_2$  are the incomes of spouses. If the individual incomes are distributed exponentially  $P(r) \propto \exp(-r/T_r)$ , then

$$P_2(r) = \int_0^r dr' P(r') P(r - r') = cr \exp(-r/T_r), \quad (22)$$

<sup>12</sup> A similar plot was constructed by Silva and Yakovenko (2005) for an earlier historical dataset.

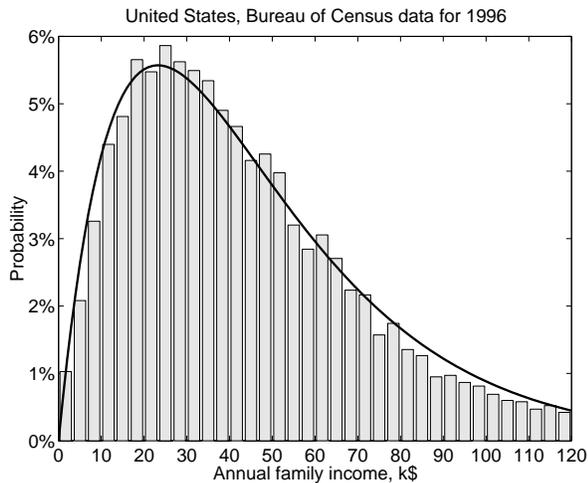


FIG. 10 *Histogram*: Probability distribution of family income for families with two adults (US Census Bureau data). *Solid line*: Fit to Eq. (22). From Drăgulescu and Yakovenko (2001a).

where  $c$  is a normalization constant. Fig. 10 shows that Eq. (22) is in good agreement with the family income distribution data from the US Census Bureau (Drăgulescu and Yakovenko, 2001a). In Eq. (22), we assumed that incomes of spouses are uncorrelated. This simple approximation is indeed supported by the scatter plot of incomes of spouses shown in Fig. 11. Each family is represented in this plot by two points  $(r_1, r_2)$  and  $(r_2, r_1)$  for symmetry. We observe that the density of points is approximately constant along the lines of constant family income  $r_1 + r_2 = \text{const}$ , which indicates that incomes of spouses are approximately uncorrelated. There is no significant clustering of points along the diagonal  $r_1 = r_2$ , i.e., no strong positive correlation of spouses' incomes.

The Gini coefficient for the family income distribution (22) was analytically calculated by Drăgulescu and Yakovenko (2001a) as  $G = 3/8 = 37.5\%$ . Fig. 12 shows the Lorenz quintiles and the Gini coefficient for 1947–1994 plotted from the US Census Bureau data (Drăgulescu and Yakovenko, 2001a). The solid line, representing the Lorenz curve calculated from Eq. (22), is in good agreement with the data. The systematic deviation for the top 5% of earners results from the upper tail, which has a less pronounced effect on family income than on individual income, because of income averaging in the family. The Gini coefficient, shown in the inset of Fig. 12, is close to the calculated value of 37.5%. Notice that income distribution is very stable for a long period of time, which was also recognized by economists (Levy, 1987). Moreover, the average  $G$  for the developed capitalist countries of North America and western Europe, as determined by the World Bank, is also close to the calculated value 37.5% (Drăgulescu and Yakovenko, 2003). However, within

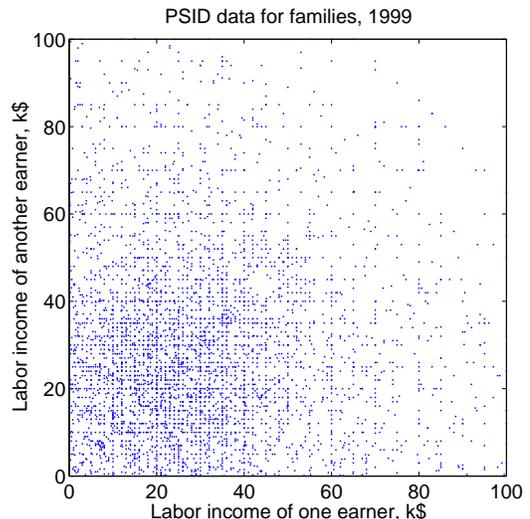


FIG. 11 Scatter plot of the spouses' incomes  $(r_1, r_2)$  and  $(r_2, r_1)$  based on the data from the Panel Study of Income Dynamics (PSID). From Drăgulescu and Yakovenko (2003).

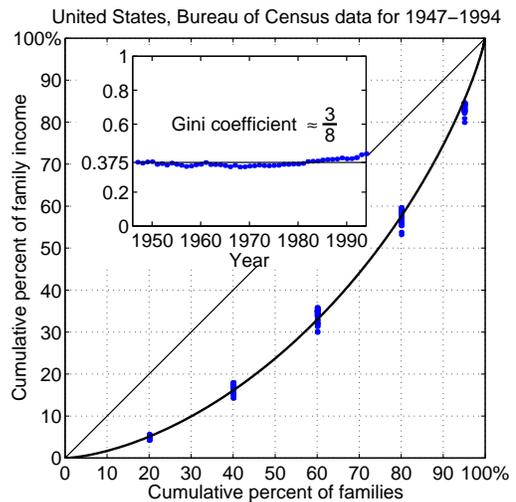


FIG. 12 *Main panel*: Lorenz plot for family income calculated from Eq. (22), compared with the US Census data points. *Inset*: The US Census data points for the Gini coefficient for families, compared with the theoretically calculated value  $3/8=37.5\%$ . From Drăgulescu and Yakovenko (2001a).

this average, nations or groups of nations may have quite different Gini coefficients that persist over time due to specific historical, political, or social circumstances (Rosser and Rosser, 2004). The Nordic economies, with their famously redistributive welfare states, have  $G$  in the mid-20%, while many of the Latin American countries have  $G$  over 50%, reflecting entrenched social patterns inherited from the colonial era.

Income distribution has been examined in econophysics papers for different countries: Japan (Aoyama *et al.*, 2003; Ferrero, 2004, 2005; Fujiwara *et al.*, 2003; Nirei and Souma,

2007; Souma, 2001, 2002; Souma and Nirei, 2005), Germany (Clementi and Gallegati, 2005a; Clementi, Gallegati, and Kaniadakis, 2007), the UK (Clementi and Gallegati, 2005a; Clementi, Gallegati, and Kaniadakis, 2007; Ferrero, 2004, 2005; Richmond, Hutzler, Coelho, and Repetowicz, 2006), Italy (Clementi, Di Matteo, and Gallegati, 2006; Clementi and Gallegati, 2005b; Clementi, Gallegati, and Kaniadakis, 2007), the USA (Clementi and Gallegati, 2005a; Rawlings *et al.*, 2004), India (Sinha, 2006), Australia (Banerjee, Yakovenko, and Di Matteo, 2006; Clementi, Di Matteo, and Gallegati, 2006; Di Matteo, Aste, and Hyde, 2004), and New Zealand (Ferrero, 2004, 2005). The distributions are qualitatively similar to the results presented in this section. The upper tail follows a power law and comprises a small fraction of population. To fit the lower part of the distribution, different papers used the exponential, Gamma, and log-normal distributions. Unfortunately, income distribution is often reported by statistical agencies for households, so it is difficult to differentiate between one-earner and two-earner income distributions. Some papers used interpolating functions with different asymptotic behavior for low and high incomes, such as the Tsallis function (Ferrero, 2005) and the Kaniadakis function (Clementi, Gallegati, and Kaniadakis, 2007). However, the transition between the lower and upper classes is not smooth for the US data shown in Figs. 6 and 7, so such functions would not be useful in this case. The special case is income distribution in Argentina during the economic crisis, which shows a time-dependent bimodal shape with two peaks (Ferrero, 2005).

## B. Theoretical models of income distribution

Having examined the empirical data on income distribution, let us now discuss theoretical models. Income  $r_i$  is the influx of money per unit time to an agent  $i$ . If the money balance  $m_i$  is analogous to energy, then the income  $r_i$  would be analogous to power, which is the energy flux per unit time. So, one should conceptually distinguish between the distributions of money and income. While money is regularly transferred from one agent to another in pairwise transactions, it is not typical for agents to trade portions of their income. Nevertheless, indirect transfer of income may occur when one employee is promoted and another demoted while the total annual budget is fixed, or when one company gets a contract whereas another one loses it, etc. A reasonable approach, which has a long tradition in the economic literature (Champernowne, 1953; Gibrat, 1931; Kalecki, 1945), is to treat individual income  $r$  as a stochastic process and study its probability distribution. In general, one can study a Markov process generated by a matrix of

transitions from one income to another. In the case where the income  $r$  changes by a small amount  $\Delta r$  over a time period  $\Delta t$ , the Markov process can be treated as income diffusion. Then one can apply the general Fokker-Planck equation (Lifshitz and Pitaevskii, 1981) to describe evolution in time  $t$  of the income distribution function  $P(r, t)$  (Silva and Yakovenko, 2005)

$$\frac{\partial P}{\partial t} = \frac{\partial}{\partial r} \left[ AP + \frac{\partial(BP)}{\partial r} \right], \quad A = -\frac{\langle \Delta r \rangle}{\Delta t}, \quad B = \frac{\langle (\Delta r)^2 \rangle}{2\Delta t}. \quad (23)$$

The coefficients  $A$  and  $B$  in Eq. (23) are determined by the first and second moments of income changes per unit time. The stationary solution  $\partial_t P = 0$  of Eq. (23) obeys the following equation with the general solution

$$\frac{\partial(BP)}{\partial r} = -AP, \quad P(r) = \frac{c}{B(r)} \exp \left( - \int^r \frac{A(r')}{B(r')} dr' \right). \quad (24)$$

For the lower part of the distribution, it is reasonable to assume that  $\Delta r$  is independent of  $r$ , i.e., the changes in income are independent of income itself. This process is called the additive diffusion (Silva and Yakovenko, 2005). In this case, the coefficients in Eq. (23) are the constants  $A_0$  and  $B_0$ . Then Eq. (24) gives the exponential distribution  $P(r) \propto \exp(-r/T_r)$  with the effective income temperature  $T_r = B_0/A_0$ .<sup>13</sup> The coincidence of this result with the Boltzmann-Gibbs exponential law (1) and (7) is not accidental. Indeed, instead of considering pairwise interaction between particles, one can derive Eq. (1) by considering energy transfers between a particle and a big reservoir, as long as the transfer process is “additive” and does not involve a Maxwell-demon-like discrimination (Basu and Mohanty, 2008). Although money and income are different concepts, they may have similar distributions, because they are governed by similar mathematical principles. It was shown explicitly by Cordier, Pareschi, and Toscani (2005); Drăgulescu and Yakovenko (2000); Slanina (2004) that the models of pairwise money transfer can be described in a certain limit by the Fokker-Planck equation.

On the other hand, for the upper tail of income distribution, it is reasonable to expect that  $\Delta r \propto r$ , i.e., income changes are proportional to income itself. This is known as the proportionality principle of Gibrat (1931), and the process is called the multiplicative diffusion (Silva and Yakovenko, 2005). In this case,  $A = ar$  and  $B = br^2$ , and Eq. (24) gives the power-law distribution  $P(r) \propto 1/r^{\alpha+1}$  with  $\alpha = 1 + a/b$ .

Generally, the lower-class income comes from wages and salaries, where the additive process is appropriate, whereas the upper-class income comes from bonuses, investments, and capital gains, calculated in percentages, where the multiplicative process applies (Milaković,

<sup>13</sup> Notice that a meaningful stationary solution (24) requires that  $A > 0$ , i.e.,  $\langle \Delta r \rangle < 0$ .

2005). However, the additive and multiplicative processes may coexist. An employee may receive a cost-of-living raise calculated in percentages (the multiplicative process) and a merit raise calculated in dollars (the additive process). Assuming that these processes are uncorrelated, we have  $A = A_0 + ar$  and  $B = B_0 + br^2 = b(r_0^2 + r^2)$ , where  $r_0^2 = B_0/b$ . Substituting these expressions into Eq. (24), we find

$$P(r) = c \frac{e^{-(r_0/T_r) \arctan(r/r_0)}}{[1 + (r/r_0)^2]^{1+a/2b}}. \quad (25)$$

The distribution (25) interpolates between the exponential law for low  $r$  and the power law for high  $r$ , because either the additive or the multiplicative process dominates in the corresponding limit. The crossover between the two regimes takes place at  $r = r_0$ , where the additive and multiplicative contributions to  $B$  are equal. The distribution (25) has three parameters: the “income temperature”  $T_r = A_0/B_0$ , the Pareto exponent  $\alpha = 1 + a/b$ , and the crossover income  $r_0$ . It is a minimal model that captures the salient features of the empirical income distribution. Eq. (25) was obtained by Yakovenko (2009), and a more general formula for correlated additive and multiplicative processes was derived by Fiaschi and Marsili (2009) for a sophisticated economic model. Fits of the IRS data using Eq. (25) are shown in Fig. 13 reproduced from Banerjee (2008). A mathematically similar, but more economically oriented, model was proposed by Nirei and Souma (2007); Souma and Nirei (2005), where labor income and assets accumulation are described by the additive and multiplicative processes correspondingly. A general stochastic process with additive and multiplicative noise was studied numerically by Takayasu *et al.* (1997), but the stationary distribution was not derived analytically. A similar process with discrete time increments was studied by Kesten (1973). Besides economic applications, Eq. (25) may be also useful for general stochastic processes with additive and multiplicative components.

To verify the multiplicative and additive hypotheses empirically, it is necessary to have data on income mobility, i.e., the income changes  $\Delta r$  of the same people from one year to another. The distribution of income changes  $P(\Delta r|r)$  conditional on income  $r$  is generally not available publicly, although it can be reconstructed by researchers at the tax agencies. Nevertheless, the multiplicative hypothesis for the upper class was quantitatively verified by Aoyama *et al.* (2003); Fujiwara *et al.* (2003) for Japan, where such data for the top taxpayers are publicly available.

The power-law distribution is meaningful only when it is limited to high enough incomes  $r > r_0$ . If all incomes  $r$  from 0 to  $\infty$  follow a purely multiplicative process ( $A_0 = 0$  and  $B_0 = 0$ ), then one can change to a logarithmic variable  $x = \ln(r/r_*)$  in Eq. (23) and show that it gives a Gaussian time-dependent distribution  $P_t(x) \propto \exp(-x^2/2\sigma^2 t)$  for  $x$ , i.e., the log-normal distribution for  $r$ , also known as the Gibrat distribu-

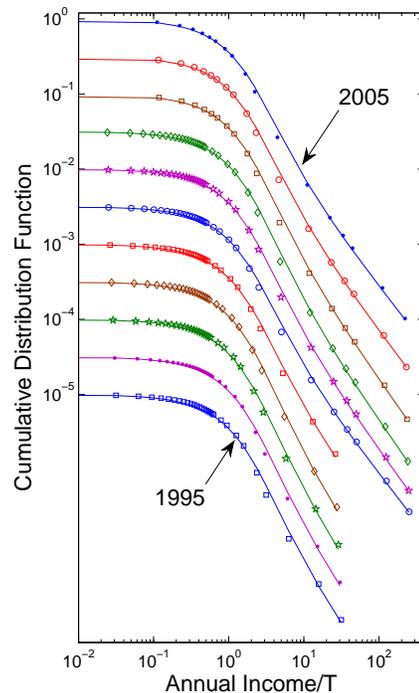


FIG. 13 Fits of the IRS data for income distribution using Eq. (25). Plots for different years are shifted vertically for clarity. From Banerjee (2008).

tion (Gibrat, 1931). However, the width of this distribution increases in time, so the distribution is not stationary. This was pointed out by Kalecki (1945) a long time ago, but the log-normal distribution is still widely used for fitting income distribution, despite this fundamental logical flaw in its justification. In the classic paper, Champernowne (1953) showed that a multiplicative process gives a stationary power-law distribution when a boundary condition is imposed at  $r_0 \neq 0$ . Later, this result was rediscovered by econophysicists (Levy, 2003; Levy and Solomon, 1996; Sornette and Cont, 1997). In Eq. (25), the exponential distribution of the lower class effectively provides such a boundary condition for the power law of the upper class. Notice also that Eq. (25) reduces to Eq. (18) in the limit  $r_0 \rightarrow 0$  with  $B_0 = 0$ , but  $A_0 \neq 0$ .

There are alternative approaches to income distribution in economic literature. One of them, proposed by Lydall (1959), involves social hierarchy. Groups of people have leaders, which have leaders of the higher order, and so on. The number of people decreases geometrically (exponentially) with the increase in the hierarchical level. If individual income increases by a certain factor (i.e., multiplicatively) when moving to the next hierarchical level, then income distribution follows a power law (Lydall, 1959). However, this original argument of Lydall can be easily modified to produce the exponential distribution. If individual income increases by a certain amount, i.e., income increases linearly with the hierarchi-

cal level, then income distribution is exponential. The latter process seems to be more realistic for moderate annual incomes below 100 k\$. A similar scenario is the Bernoulli trials (Feller, 1966), where individuals have a constant probability of increasing their income by a fixed amount. We see that the deterministic hierarchical models and the stochastic models of additive and multiplicative income mobility represent essentially the same ideas.

## V. CONCLUSIONS

The “invasion” of physicists into economics and finance at the turn of the millennium is a fascinating phenomenon. It generated a lively public debate about the role and future perspectives of econophysics, covering both theoretical and empirical issues.<sup>14</sup> The econophysicist Joseph McCauley proclaimed that “Econophysics will displace economics in both the universities and boardrooms, simply because what is taught in economics classes doesn’t work” (Ball, 2006). Although there is some truth in his arguments (McCauley, 2006), one may consider a less radical scenario. Econophysics may become a branch of economics, in the same way as game theory, psychological economics, and now agent-based modeling became branches of economics. These branches have their own interests, methods, philosophy, and journals. When infusion of new ideas from a different field happens, the main contribution often consists not in answering old questions, but in raising new questions. Much of the misunderstanding between economists and physicists happens not because they are getting different answers, but because they are answering different questions.

The subject of income and wealth distributions and social inequality was very popular at the turn of another century and is associated with the names of Pareto, Lorenz, Gini, Gibrat, and Champernowne, among others. Following the work by Pareto, attention of researchers was primarily focused on the power laws. However, when physicists took a fresh look at the empirical data, they found a different, exponential law for the lower part of the distribution. Demonstration of the ubiquitous nature of the exponential distribution for money, wealth, and income is one of the new contributions produced by econophysics.<sup>15</sup> The motivation, of course, came from the Boltzmann-Gibbs distribution in physics. Further studies revealed a more detailed picture of the two-class

distribution in a society. Although social classes have been known in political economy since Karl Marx, realization that they are described by simple mathematical distributions is quite new. Very interesting work was done by the computer scientist Ian Wright (2005, 2009), who demonstrated emergence of two classes in an agent-based simulation of initially equal agents. This work has been further developed in the upcoming book by Cottrell, Cockshott, Michaelson, Wright, and Yakovenko (2009), integrating economics, computer science, and physics.

Econophysics may be also useful for teaching of statistical physics. If nothing else, it helps to clarify the foundations of statistical physics by applying it to non-traditional objects. Practitioners of statistical physics know very well that the major fascinating attraction of this field is the enormous breadth of its applications.

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<sup>14</sup> See, for example, Anglin (2005); Ball (2006); Carbone *et al.* (2007); Gallegati, Keen, Lux, and Ormerod (2006); Keen (2008); Lux (2005, 2009); McCauley (2006); Richmond, Chakrabarti, Chatterjee, and Angle (2006); Rosser (2006, 2008b); Stauffer (2004); Yakovenko (2009).

<sup>15</sup> The exponential distribution is also ubiquitous in the probability distributions of financial returns (Kleinert and Chen, 2007; McCauley and Gunaratne, 2003; Silva, Prange, and Yakovenko, 2004) and the growth rates of firms.

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