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AN OPTIMAL CONTROL PROBLEM BY PARABOLIC EQUATION WITH BOUNDARY SMOOTH CONTROL AND AN INTEGRAL CONSTRAINT

Alexander Arguchintsev

Institute of Mathematics, Economics and Computer Science
Irkutsk State University, Irkutsk, Russia

Vasilisa Poplevko *

Institute of Mathematics, Economics and Computer Science
Irkutsk State University, Irkutsk, Russia

Abstract. In the paper, we consider an optimal control problem by differential boundary condition of parabolic equation. We study this problem in the class of smooth controls satisfying certain integral constraints. For the problem under consideration we obtain a necessary optimality condition and propose a method for improving admissible controls. For illustration, we solve one numerical example to show the effectiveness of the proposed method.

1. **Introduction.** Problems of optimal control by boundary condition of parabolic equation arise in modeling of processes of the thermal conductivity, diffusion, filtering [3, 4, 5, 6, 7, 10]. In particular, such problems describe mass transfer processes in column-type apparatuses, taking into account the longitudinal mixing. Control functions in these problems represent flows of raw materials or finished products [5, 6].

A fundamental difference from classical statements of optimal control problems [8, 9] is the investigation of problems in the class of smooth admissible controls and classical solutions [11] of initial-boundary problems.

In this paper, we study the problem in the class of smooth controls that satisfy the integral constraint. It is impossible to use optimal control methods based on Pontryagin's maximum principle. These methods are focused on the classes of discontinuous controls. We apply approach [1, 2] which is based on using a special

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* Corresponding author: Vasilisa Poplevko.

variation that provides smoothness of controls and satisfaction the integral constraint.

The paper is organised as follows. Section 2 formulates the problem for the paper. In the following section, we obtain an increment formula for the cost functional. Section 4 is devoted to the necessary optimality condition. In Section 5, we describe a general scheme of the method which is based on the optimality condition. A numerical experiment is presented for illustration in Section 6 and the paper ends with some concluding remarks in Section 7.

2. **Problem statement.** Consider the equation

$$x_t - x_{ss} = f(s,t), \tag{1}$$

$$\Pi = S \times T,\, S = [s_0, s_1], T = [t_0, t_1],$$

where $x = x(s,t)$ is a state function.

Initial-boundary conditions take the form

$$x(s,t_0) = x^0(s), \quad s \in S; \quad x_s(s_1,t) = q(t),$$

$$x_t(s_0,t) = g(x(s_0,t), u(t), t), \qquad t \in T. \tag{2}$$

Here $u(t)$ is a smooth control on segment T and satisfies the integral constraint

$$\int\limits_T \Phi(u(t))\, dt = M, \tag{3}$$

where M is a constant and function Φ satisfies the following condition

$$\Phi(\lambda u) = \lambda^\alpha \Phi(u), \quad \alpha \geq 1.$$

The problem is to minimize the functional

$$J(u) = \int\limits_S \varphi(x(s,t_1),s)\, ds + \iint\limits_\Pi F(x,s,t)\, ds\, dt, \tag{4}$$

defined on the solutions of the problem (1), (2) under admissible control functions (3).

We study the problem (1) – (4) under the following suppositions:

1) functions $f(s,t)$, $x^0(s)$, $q(t)$ are continuous with respect to their arguments on the set Π, S, and T, respectively;

2) functions $F(x,s,t)$ and $\varphi(x,s)$ are continuous with respect to their arguments, and they have continuous and bounded partial derivatives with respect to state function x;

3) function $g(x,u,t)$ is continuous and continuously differentiable, and has bounded partial derivatives with respect to x and u.

We understand the solution to problem (1), (2) that corresponds to control (3) in the classical sense (as continuous and continuously differentiable one)[11].

3. **Increment formula for the cost functional.** Consider two admissible processes, namely, the initial process $\{u, x\}$ and the perturbated one $\{\widetilde{u} = u + \Delta u, \widetilde{x} = x + \Delta x\}$. Define $Dx = x_t - x_{ss}$. Then the problem is written in the following form

$$D\Delta x = 0,$$

$$\Delta x(s, t_0) = 0, \quad s \in S; \quad \Delta x_s(s_1, t) = 0, \quad t \in T,$$

$$\Delta x_t(s_0, t) = \Delta g(x(s_0, t), u(t), t), \tag{5}$$

$$\Delta J(u) = J(\widetilde{u}) - J(u) = \int\limits_S \Delta\varphi(x(s, t_1), s)\, ds + \iint\limits_\Pi \Delta F(x, s, t)\, ds\, dt.$$

Add the following terms to the increment formula for the cost functional to obtain Lagrangian functional

$$\iint\limits_\Pi \psi(s, t) D\Delta x\, ds\, dt,$$

$$\int\limits_T p(t)[\Delta x_t(s_0, t) - \Delta g(x(s_0, t), u(t), t)]\, dt.$$

Applying integration by parts, we have

$$\begin{aligned}
\Delta J(u) = &\int\limits_S \Delta\varphi(x(s, t_1), s)\, ds + \iint\limits_\Pi \Delta F(x, s, t)\, ds\, dt \\
&+ \int\limits_S [\psi(s, t_1)\Delta x(s, t_1) - \psi(s, t_0)\Delta x(s, t_0)]\, ds - \iint\limits_\Pi \psi_t \Delta x\, ds\, dt \\
&- \int\limits_T [\psi(s_1, t)\Delta x_s(s_1, t) - \psi(s_0, t)\Delta x_s(s_0, t) - \psi_s \Delta x(s_1, t) + \psi_s \Delta x(s_0, t)]\, dt \\
&- \iint\limits_\Pi \psi_{ss}\Delta x\, ds dt + p(t_1)\Delta x(s_0, t_1) - p(t_0)\Delta x(s_0, t_0) \\
&- \int\limits_T p_t \Delta x(s_0, t)\, dt - \int\limits_T p(t)\Delta g\, dt.
\end{aligned}$$

Introduce the following auxiliary function

$$h(p(t), x(s_0, t), u(t), t) = p(t) \cdot g(x(s_0, t), u(t), t).$$

Then

$$\Delta h(p, x, u, t) = h(p, \widetilde{x}, \widetilde{u}, t) - h(p, x, u, t).$$

Add and subtract the following term $h(p, x, \widetilde{u}, t)$. Then, we get

$$\Delta h(p, x, u, t) = \Delta h_{\widetilde{u}}(p, x, u, t) + \Delta h_{\widetilde{x}}(p, x, \widetilde{u}, t).$$

Use the following expansions

$$\Delta\varphi(x(s, t_1), s) = \frac{\partial\varphi(x(s, t_1), s)}{\partial x}\Delta x(s, t_1) + o_\varphi(|\Delta x(s, t_1)|),$$

$$\Delta F(x, s, t) = \frac{\partial F(x, s, t)}{\partial x}\Delta x(s, t) + o_F(|\Delta x(s, t)|),$$

$$\Delta_{\widetilde{x}} h(p, x(s_0,t), \widetilde{u}, t) = \frac{\partial h(p, x(s_0,t), \widetilde{u}, t)}{\partial x} \Delta x(s_0,t) + o_h(|\Delta x(s_0,t)|).$$

Let functions $\psi(s,t),\ p(t)$ be solutions to the following adjoint problem (a first-order necessary optimality conditions with respect to state function for Lagrangian)

$$\begin{aligned} D^*\psi = &F_x(x,s,t), \quad \psi(s,t_1) = -\varphi_x(x(s,t_1),s), \\ &\psi(s_0,t) = 0, \quad \psi_s(s_1,t) = 0; \\ &p_t = -h_x - \psi_s(s_0,t), \quad p(t_1) = 0, \end{aligned} \tag{6}$$

where $D^*\psi = \psi_t + \psi_{ss}$.

Then, the increment formula for the functional takes the form

$$\begin{aligned} J(u) = &-\int_T [\varphi_x(x(s,t_1),s) + \psi(s,t_1)]\Delta x(s,t_1) + o_\varphi(|\Delta x(s,t_1)|)\, dt \\ &- \iint_\Pi [\psi_t + \psi_{ss} - F_x]\Delta x(s,t) - (o_F(|\Delta x(s,t)|)\, dsdt \\ &- \int_T \psi_s \Delta x(s_1,t)\, dt + p(t_1)\Delta x(s_0,t_1) \\ &- \int_T \Delta_{\widetilde{u}} h(p(t), x(s_0,t), u(t), t)\, dt - \int_T [p_t + \psi_s(s_0,t) \\ &+ h_x(p, x(s_0,t), \widetilde{u}, t)]\Delta x(s_0,t) + o_h(|\Delta x(s,t_0)|)\, dt. \end{aligned}$$

Transform the term

$$\begin{aligned} &h_x(p, x(s_0,t), \widetilde{u}, t) = h_x(p, x(s_0,t), \widetilde{u}, t) \pm h_x(p, x(s_0,t), u, t) \\ &= h_x(p, x(s_0,t), u, t) + \Delta_{\widetilde{u}} h_x(p, x(s_0,t), u, t). \end{aligned}$$

Then, we get

$$\Delta J(u) = -\int_T \Delta_{\widetilde{u}} h(p(t), x(s_0,t), u(t), t)\, dt + \eta. \tag{7}$$

Here

$$\begin{aligned} \eta = &\int_S o_\varphi(|\Delta x(s,t_1)|)\, ds + \iint_\Pi (o_F(|\Delta x(s,t)|)\, dsdt \\ &- \int_T [o_h(|\Delta x(s_0,t)|) + \Delta_{\widetilde{u}} h_x(p(t), x(s_0,t), u(t), t) \cdot \Delta x(s_0,t)]\, dt. \end{aligned} \tag{8}$$

Lemma 3.1. *If condition (5) is valid the estimation of a state increment (analogously to [12]) takes the form*

$$\int_S |\Delta x(s,t_1)|^2\, ds \le K(|\Delta u|^2). \tag{9}$$

Here

$$\begin{aligned} K(|\Delta u|^2) = &(\frac{1}{\varepsilon_1} L_1^2 + \frac{1}{\varepsilon_2} L_1^2 L (t_1 - t_0)^2 + \frac{1}{\varepsilon_2} L_1^2) \iint_\Pi \Delta u^2\, dsdt \\ &+ \varepsilon_2 (s_1 - s_0)(t_1 - t_0) L_1^2 \int_T \Delta u^2 dt, \end{aligned}$$

where $\varepsilon_1 > 0,\ \ \varepsilon_2 > 0;\ \ L > 0,\ \ L_1 > 0$ *(L is Lipschitz constant,* $L_1 = Le^{L(t_1-t_0)}$*).*

Proof. To get (9), we consider the following trivial equality

$$
\begin{aligned}
0 = &\iint\limits_{\Pi} (\Delta x_t - \Delta x_{ss})(\Delta x(s,t) - 2\Delta x(s_0,t))\, dsdt = \iint\limits_{\Pi} \Delta x_t \Delta x(s,t)\, dsdt \\
&- \iint\limits_{\Pi} \Delta x_{ss}\Delta x(s,t)\, dsdt - 2\iint\limits_{\Pi} \Delta x_t \Delta x(s_0,t)\, dsdt + 2\iint\limits_{\Pi} \Delta x_{ss}\Delta x(s_0,t)\, dsdt \\
&= \int\limits_{S} \Delta x^2(s,t)|_{t=t_0}^{t=t_1}\, ds - 2\int\limits_{T} \Delta x_s \Delta x(s,t)|_{s=s_0}^{s=s_1}\, dt + 2\iint\limits_{\Pi} \Delta x_s^2\, dsdt \\
&- 2\int\limits_{S} \Delta x(s,t)\Delta x(s_0,t)|_{t=t_0}^{t=t_1}\, ds + 2\iint\limits_{\Pi} \Delta x \Delta x_t(s_0,t)\, dsdt \\
&+ 2\int\limits_{T} \Delta x_s \Delta x(s_0,t)|_{s=s_0}^{s=s_1}\, dt - 2\iint\limits_{\Pi} \Delta x_s(\Delta x(s_0,t))_s\, dsdt.
\end{aligned}
$$

We obtain

$$
\begin{aligned}
&\frac{1}{2}\int\limits_{S} \Delta x^2(s,t_1)\, ds + \iint\limits_{\Pi} \Delta x_s^2\, dsdt \\
&\qquad = \int\limits_{S} \Delta x(s,t_1)\Delta x(s_0,t_1)\, ds - \iint\limits_{\Pi} \Delta x(s,t)\Delta g\, dsdt.
\end{aligned}
\tag{10}
$$

Estimate the right side in (10)

$$
\begin{aligned}
&|\int\limits_{S} \Delta x(s,t_1)\Delta x(s_0,t_1)\, ds - \iint\limits_{\Pi} \Delta x(s,t)\Delta g\, dsdt| \\
&\qquad \le \int\limits_{S} |\Delta x(s,t_1)\Delta x(s_0,t_1)|\, ds + \iint\limits_{\Pi} |\Delta x(s,t)\Delta g|\, dsdt.
\end{aligned}
$$

Further, we use the following inequality

$$(a+b)^2 \le 2a^2 + 2b^2,$$

for any $a,\ b$ and Young's inequality

$$|ab| \le \frac{1}{2}\varepsilon a^2 + \frac{1}{2\varepsilon} b^2,$$

for any $a,\ b,\ \varepsilon > 0$. Then, we obtain

$$
\begin{aligned}
|\Delta x(s,t_1)\Delta x(s_0,t_1)| &\le \frac{1}{2}\varepsilon_1 \Delta x^2(s,t_1) + \frac{1}{2\varepsilon_1}\Delta x^2(s_0,t_1), \quad \varepsilon_1 > 0, \\
|\Delta x(s,t)\Delta g| &\le \frac{1}{2}\varepsilon_2 \Delta x^2(s,t) + \frac{1}{2\varepsilon_2}(\Delta g)^2, \quad \varepsilon_2 > 0.
\end{aligned}
\tag{11}
$$

Since

$$
\begin{aligned}
\Delta x^2(s,t) = &(\int\limits_{s_0}^{s} \Delta x_s(\xi,t)\, d\xi + \Delta x(s_0,t))^2 \le 2(\int\limits_{s_0}^{s} \Delta x_s(\xi,t)\, d\xi)^2 + 2\Delta x^2(s_0,t) \\
&\le 2(s_1 - s_0)\int\limits_{S} \Delta x_s^2(\xi,t)\, d\xi + 2\Delta x^2(s_0,t), \quad (s,t) \in \Pi,
\end{aligned}
$$

then

$$\iint\limits_{\Pi} \Delta x^2(s,t)\,dsdt \le 2(s_1-s_0)^2 \iint\limits_{\Pi} \Delta x_s^2(s,t)\,dsdt + 2(s_1-s_0)\int\limits_T \Delta x^2(s_0,t)\,dt. \tag{12}$$

Taking into account the assumption of the problem for function $g(x(s_0,t),u(t),t)$, we get

$$|\Delta x(s_0,t)| \le \int\limits_{t_0}^{t} (L|\Delta x(s_0,t)| + |\Delta u|)\,d\tau \le L_1 \int\limits_{t_0}^{t} |\Delta u|\,d\tau,\ L>0,\quad L_1 = Le^{L(t_1-t_0)},$$

where L is Lipschitz constant;

$$|\Delta x(s_0,t)|^2 \le L_1^2(t_1-t_0)\int\limits_T |\Delta u|^2\,dt. \tag{13}$$

Using (11), (12), (13) in (10), we obtain

$$(\frac{1}{2}-\frac{1}{2}\varepsilon_1)\int\limits_S |\Delta x(s,t_1)|^2\,ds + (1-\varepsilon_2(s_1-s_0)^2)\iint\limits_{\Pi} |\Delta x_s|^2\,dsdt \le (\frac{1}{\varepsilon_1}L_1^2 + \frac{1}{\varepsilon_2}L_1^2 L(t_1-t_0)^2 + \frac{1}{\varepsilon_2}L_1^2)\iint\limits_{\Pi} |\Delta u|^2\,dsdt + \varepsilon_2(s_1-s_0)(t_1-t_0)L_1^2\int\limits_T |\Delta u|^2\,dt. \tag{14}$$

Take numbers $\varepsilon_1,\ \varepsilon_2$ such that $(\frac{1}{2}-\frac{1}{2}\varepsilon_1)>0$, $(1-\varepsilon_2(s_1-s_0)^2)>0$. Then, from (14), we get estimate (9). □

4. **Necessary optimality condition.** Since admissible controls belong to the class of smooth functions, we apply the idea of the general approach [1, 2] based on using a special variation that provides smoothness of control and satisfaction the integral constraint. The varied control obeys the formula

$$u_{\varepsilon,\delta}(t) = \lambda(t)u(t+\varepsilon\delta(t))\quad \lambda(t) = (1+\varepsilon\dot{\delta}(t))^{\frac{1}{\alpha}},\quad t\in T, \tag{15}$$

where $\varepsilon\in[0,1]$ is a parameter of variation, $\delta(t)$ is a twice continuously differentiable function and satisfies the following conditions

$$t_0 \le t+\delta(t) \le t_1,\quad t\in T. \tag{16}$$

$$\delta(t_0)=\delta(t_1)=0,$$

$$|\dot{\delta}(t)|\le 1,\quad t\in T. \tag{17}$$

Show that control $u_{\varepsilon,\delta}(t)$ (15) satisfies integral constraint (3).

$$\int\limits_T \Phi(u_{\varepsilon,\delta}(t))\,dt = \int\limits_T \Phi(\lambda(t)u(t+\varepsilon\delta(t)))\,dt = \int\limits_T \lambda^{\alpha}(t)\Phi(u(t+\varepsilon\delta(t)))\,dt = \int\limits_T (1+\varepsilon\dot{\delta}(t))\Phi(u(t+\varepsilon\delta(t)))\,dt.$$

Denote $\xi = t + \varepsilon\delta(t)$. Then $d\xi = (1 + \varepsilon\dot{\delta}(t))\,dt$ and

$$\int\limits_T \Phi(u_{\varepsilon,\delta}(t))\,dt = \int\limits_T \Phi(u(\xi))\,d\xi = M.$$

To decompose the function $(1 + \varepsilon\dot{\delta}(t))^{\frac{1}{\alpha}}$ in a row on degrees ε:

$$\begin{aligned}(1 + \varepsilon\dot{\delta}(t))^{\frac{1}{\alpha}} &= 1 + \frac{1}{\alpha}\varepsilon\dot{\delta}(t) + \frac{1}{2!}\frac{1}{\alpha}(\frac{1}{\alpha} - 1)\varepsilon^2\dot{\delta}^2(t) + ...\\ ... &+ \frac{1}{n!}\frac{1}{\alpha}(\frac{1}{\alpha} - 1)\cdot(\frac{1}{\alpha} - n + 1)\varepsilon^n\dot{\delta}^n(t) + ...\end{aligned}$$

Taking into account that $\varepsilon \in [0, 1]$, for convergence of the row (under $\alpha > 1$) implementation of condition (17) is required. Then the increment formula for the control function takes the form

$$\begin{aligned}\Delta u(t) =& u_\varepsilon(t) - u(t) = \lambda(t)u(t + \varepsilon\delta(t)) - u(t)\\ =&(1 + \varepsilon\dot{\delta}(t))^{\frac{1}{\alpha}}u(t + \varepsilon\delta(t)) - u(t)\\ =&[1 + \frac{1}{\alpha}\varepsilon\dot{\delta}(t) + \frac{1}{2!}\frac{1}{\alpha}(\frac{1}{\alpha} - 1)\varepsilon^2\dot{\delta}^2(t) + ...]u(t + \varepsilon\delta(t)) - u(t)\\ =&u(t + \varepsilon\delta(t)) - u(t) + \frac{1}{\alpha}\varepsilon\dot{\delta}(t)u(t + \varepsilon\delta(t)) + o(\varepsilon)\\ =&\dot{u}(t)\varepsilon\delta(t) + \frac{1}{\alpha}u(t)\varepsilon\dot{\delta}(t) + o(\varepsilon).\end{aligned} \tag{18}$$

Then, from (7),(8) and using (18), we get

$$\begin{aligned}\Delta J(u) =& -\int\limits_T (h_u \cdot \Delta u)\,dt + \eta\\ =& -\int\limits_T h_u \cdot (\dot{u}(t)\varepsilon\delta(t) + \frac{1}{\alpha}u(t)\varepsilon\dot{\delta}(t) + o(\varepsilon))\,dt + \eta\\ =& -\varepsilon\int\limits_T (h_u \cdot \dot{u}(t) \cdot \delta(t))\,dt - \varepsilon\frac{1}{\alpha}\int\limits_T (h_u \cdot u(t) \cdot \dot{\delta}(t))\,dt + \eta_1\\ =& -\varepsilon\int\limits_T (h_u \cdot \dot{u}(t) \cdot \delta(t))\,dt - \varepsilon\frac{1}{\alpha}(h_u \cdot u(t) \cdot \delta(t))|_{t=t_0}^{t=t_1}\\ &+ \varepsilon\frac{1}{\alpha}\int\limits_T (h_u \cdot u(t))_t\delta(t))\,dt + \eta_1,\end{aligned}$$

where

$$\eta_1 = \eta - \int\limits_T (h_u \cdot o(\varepsilon))\,dt.$$

Since $\delta(t_0) = \delta(t_1) = 0$ and $\eta_1 \sim o(\varepsilon)$, we get

$$\Delta J(u) = -\varepsilon\int\limits_T (h_u \cdot \dot{u}(t) - \frac{1}{\alpha}(h_u \cdot u)_t) \cdot \delta(t)\,dt + o(\varepsilon). \tag{19}$$

Increment formula for the functional (19) enables us to formulate a necessary optimality condition (analogously to results obtained in [1, 2]).

Theorem 4.1. *Let $\{u, x\}$ be the optimal process in the problem, then the following condition holds*

$$\omega(t) = h_u(p(t), x(s_0, t), u(t), t) \cdot \dot{u} - \frac{1}{\alpha}(h_u(p(t), x(s_0, t), u(t), t) \cdot u)_t = 0, \ t \in T,$$

where $p(t)$ is a solution to the adjoint problem (6).

The convergence result is given in [1, 2]. Study the problem (1) – (4) under additional suppositions:
1. functional $J(u)$ is bounded from bellow on set of admissible functions;
2. functions $F_x(x, s, t)$ and $\varphi_x(x, s)$ satisfy Lipschitz conditions with respect to state function x;
3. function $g_u(x, u, t)$ satisfies Lipschitz conditions with respect to function u.

Then for any admissible initial approximation the method generates a sequence which is relaxation one

$$J(u^{k+1}) < J(u^k), \quad k = 0, 1, 2, ...,$$

and converge in the sense

$$\mu(u^k) = \int_T \delta_k(t)\omega_k(t)\, dt \to 0, \quad k \to \infty.$$

5. **Method for improvement of admissible control.** Let us describe the general scheme of the method based on the use of the stated optimality condition [1, 2].
1) Let $u^k(t)$ be an admissible control calculated on the k-th iteration.
2) For the control $u^k(t)$ find functions $x^k(s, t)$ and $\psi^k(s, t)$, $p^k(t)$ which are the solutions to problems (1), (2) and (6), respectively.
3) Using obtained solutions, calculate the value of the functional $J^k = J(u^k)$ and construct the function

$$\omega_k(t) = h_u(p^k(t), x^k(s_0, t), u^k(t), t) \cdot \dot{u}^k(t) - \frac{1}{\alpha}(h_u(p^k(t), x^k(s_0, t), u^k(t), t) \cdot u^k)_t$$

that can be consider as a discrepancy of the fulfillment of the optimality condition. If $\omega_k(t) = 0$ then the control function u^k satisfies the optimality condition and the iteration process finishes.
4) Let $\omega_k(t) \neq 0$, consider a smooth variation of $u^k(t)$

$$u^k_{\varepsilon_k}(t) = (1 + \varepsilon_k \dot{\delta}_k(t))^{\frac{1}{\alpha}} u^k(t + \varepsilon_k \delta_k(t)),$$

$$\delta_k(t) = \frac{\gamma_k(t)}{M_k},$$

$$\gamma_k(t) = \frac{(t - t_0)(t_1 - t)\omega_k(t)}{(t_1 - t_0) \max\limits_{t \in T} |\omega_k(t)|},$$

$$M_k = \max_{t \in T} |\dot{\gamma}_k(t)|.$$

Here parameter ε_k ia a solution to the one-dimensional minimization problem

$$\varepsilon_k : \quad J(u^k_\varepsilon) \to \min, \quad \varepsilon \in [0, 1].$$

If the calculated value of this parameter is close to zero, then there is no improvement of the functional on the method step.
5) The next approximation is given by the formula

$$u^{k+1}(t) = u^k_{\varepsilon_k}(t).$$

The stop criterion consists in the fulfillment (on some k-th iteration of the method) of one of the following conditions.
a) The function $u^k(t)$ satisfies (with a given accuracy) the necessary optimality condition. For example, the value of the function $\omega_k(t)$ at each point $t \in T$ is close to zero if $\max_{t\in T} |\omega_k(t)| \leq 10^{-5}$.
b) The value of the functional calculated on the previous iteration (that with the number $k-1$) is not improved, for example, $J^k - J^{k-1} > 10^{-6}$.

6. **Numerical experiment.** Consider the application of the described method to one test example. The presented computational results are obtained by using Matlab 7.0. In the square $[0,1]\times[0,1]$ we consider the optimal control problem

$$x_t - x_{ss} = e^s \cos t, \quad s \in [0,1], \quad t \in [0,1],$$

$$x(s,0) = s + 0,3, \quad x_s(1,t) = 0,$$

$$x_t(0,t) = x(0,t)u(t), \quad \int\limits_T u(t)\,dt = 2.$$

The cost functional takes the form

$$J(u) = \frac{1}{2}\int\limits_S (x(s,t_1) - x^*(s))^2\,ds,$$

where $x^*(s) = x^*(s,t_1)$ is evaluated for the control $u^*(t) = 4(1-t)$. We solved the problem by the described method under different initial approximations:

1) The initial control is $u^0(t) = 2(\sin 2\pi t + 1)$. The value of the functional is $J(u^0) = 1.9193$.

We have obtained the following results: the value of the cost functional on the procedure output is $J(u^k) = 0.00008938$, the optimality error is $\max_{t\in T} |\omega_k(t)| = 0.0004261$, the total number of iteration equals 24, the stop criterion consisted in attaining the given accuracy with respect to the functional value.

2) The initial control is $u^0(t) = 2$. The functional value is $J(u^0) = 1.3247$.

We have obtained the following results: $J(u^k) = 0.0064837$, $\max_{t\ inT} |\omega_k(t)| = 0.02791$, the total number of iteration equals 35. The stop criterion is $\varepsilon_k \leq 10^{-5}$ (there is no improvement of the functional on the method step).

In the case of integral constraints the proposed variation for control function allows to create new values of control. But choosing not constant initial approximation we obtained better results with respect to discrepancy of the fulfillment of the optimality condition and value of the functional.

Also the problem was considered under pointwise constraints for control function. In that case numerical results depended on initial approximation too. It was necessary to choose such initial approximation that ranges over a set U. In the situation we did not get new values of control functions. We had re-sorted existing ones.

7. **Conclusion.** In this paper, we proposed the necessary optimality condition in the class of smooth control. We applied approach [1, 2] which is based on using a special variation that provides smoothness of control and satisfaction the integral constraint. The series of numerical experiments is carried out. Numerical experiments showed that computational results depend on choosing initial approximation. This conclusion was not proved by theoretical reasoning (form of initial approximation). The numerical experiments show that the proposed method of improving a smooth control can be effectively used to solve this class of problem.

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E-mail address: arguch@math.isu.ru
E-mail address: vasilisa@math.isu.ru