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An Optimal Control Problem for a Parabolic Equation in the Class of Smooth Controls

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Abstract—We consider an optimal control problem for a parabolic equation with a differential constraint on the boundary. We study this problem in the class of smooth controls satisfying certain pointwise constraints. Such problems describe mass transfer processes in column-type apparatuses, taking into account the longitudinal mixing. Control functions in these problems represent flows of raw materials or finished products. For the problem under consideration we obtain a necessary optimality condition, propose a method for improving admissible controls, and carry out the numerical experiment.

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1. THE PROBLEM

Consider the equation

$$x_t - x_{ss} = f(s,t), \tag{1.1}$$

$$(s,t) \in \Pi,\ \ \Pi = S \times T,\ \ S = [s_0, s_1],\ \ T = [t_0, t_1],$$

where $x = x(s,t)$ is the state function.

Initial-boundary conditions take the form

$$x(s,t_0) = x^0(s),\ \ s \in S;\ \ x_s(s_1,t) = q(t),$$

$$x_t(s_0,t) = g(x(s_0,t), u(t), t),\ \ t \in T. \tag{1.2}$$

Here $u(t)$ is the control function. The set of admissible controls is the totality of smooth on T functions such that

$$u(t) \in U,\ \ t \in T, \tag{1.3}$$

where the set U is compact.

One has to find an admissible control, for which the value of the objective functional

$$J(u) = \int_S \varphi(x(s,t_1), s)\, ds + \iint_\Pi F(x,s,t)\, ds\, dt \tag{1.4}$$

is minimal.

Such problems describe the rectification process in column-type apparatuses, taking into account the longitudinal mixing. Control functions represent flows of raw materials or finished products [1].

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We study problem (1.1)–(1.4) under the following assumptions:
1) functions $f(s,t)$, $x^0(s)$, and $q(t)$ are continuous on Π, S, and T, respectively;
2) functions $F(x,s,t)$ and $\varphi(x,s)$ are continuous in totality of their arguments, while their partial derivatives in x are continuous and bounded;
3) the function $g(x,u,t)$ is continuous and continuously differentiable in its arguments and has bounded derivatives in x and u.

We understand the solution to problem (1.1), (1.2) that corresponds to control (1.3) in the classical sense (as continuous and continuously differentiable one).

2. THE INCREMENT FORMULA

Consider the increment formula for the objective functional on two admissible processes, namely, the initial process $\{u,x\}$ and the varied one $\{\widetilde{u}=u+\Delta u, \widetilde{x}=x+\Delta x\}$. Denote $Dx=x_t-x_{ss}$. Then the problem stated in terms of increments takes the form

$$D\Delta x=0,$$

$$\Delta x(s,t_0)=0,\ \ s\in S;\ \ \Delta x_s(s_1,t)=0,\ \ t\in T,$$

$$\Delta x_t(s_0,t)=\Delta g(x(s_0,t),u(t),t),\ \ \Delta x(s_0,t_0)=0, \tag{2.1}$$

while the increment of the functional $\Delta J(u)=J(\widetilde{u})-J(u)$ equals

$$\Delta J(u)=\int_S \Delta\varphi(x(s,t_1),s)\,ds+\iint_\Pi \Delta F(x,s,t)\,ds\,dt.$$

Introduce the following auxiliary function:

$$h(p(t),x(s_0,t),u(t),t)=p(t)\cdot g(x(s_0,t),u(t),t).$$

Let functions $\psi(s,t)$, $p(t)$ be solutions to the following adjoint problem:

$$D^*\psi=F_x(x,s,t),\ \ \psi(s,t_1)=-\varphi_x(x(s,t_1),s),$$

$$\psi(s_0,t)=0,\ \ \psi_s(s_1,t)=0; \tag{2.2}$$

$$p_t=-h_x-\psi_s(s_0,t),\ \ p(t_1)=0,$$

where $D^*\psi=\psi_t+\psi_{ss}$. Then the increment formula for the functional takes the form

$$\Delta J(u)=-\int\limits_T \Delta_{\widetilde{u}}h(p(t),x(s_0,t),u(t),t)\,dt+\eta.$$

Here

$$\eta=\int_S o_\varphi(|\Delta x(s,t_1)|)\,ds-\iint_\Pi(o_F(|\Delta x(s,t)|)\,ds\,dt$$
$$-\int_T[o_h(|\Delta x(s_0,t)|)+\Delta_{\widetilde{u}}h_x(p(t),x(s_0,t),u(t),t)\cdot\Delta x(s_0,t)]\,dt. \tag{2.3}$$

Condition (2.1) ensures the bound (analogously to [2])

$$\int_S|\Delta x(s,t_1)|^2\,ds\le\Phi(|\Delta u|^2). \tag{2.4}$$

Here

$$\Phi(|\Delta u|^2)=\left(\frac{2}{1-\varepsilon_1}\right)\left[\left(\frac{1}{\varepsilon_1}L_1^2+\frac{1}{\varepsilon_2}L_1^2L(t_1-t_0)^2+\frac{1}{\varepsilon_2}L_1^2\right)\iint_\Pi|\Delta u|^2\,ds\,dt\right.$$

$$+ \varepsilon_2(s_1 - s_0)(t_1 - t_0)L_1^2 \int_T |\Delta u|^2 dt \Big],$$

where $\varepsilon_1 > 0$ ($\varepsilon_1 \neq 1$), $\varepsilon_2 > 0$, $L > 0$, $L_1 > 0$.

3. THE NECESSARY OPTIMALITY CONDITION

Since admissible controls belong to the class of smooth functions, we apply the idea of the general approach [3] based on the use of nonclassical variations which ensure the smoothness of admissible controls. The varied control obeys the formula

$$u_{\varepsilon,\delta}(t) = u(t + \varepsilon\delta(t)), \ \ t \in T,$$

$\varepsilon \in [0, 1]$ is the variation parameter, $\delta(t)$ is a continuously differentiable function such that $t_0 \leq t + \delta(t) \leq t_1$, $t \in T$. Taking into account the smoothness of admissible controls, we use the expansion

$$\Delta u(t) = \dot{u}(t)\varepsilon\delta(t) + o(\varepsilon).$$

With the help of bound (2.3) we get

$$\Delta J(u) = -\varepsilon \int_T h_u \cdot \dot{u}(t) \cdot \delta(t)\, dt + o(\varepsilon).$$

The formula for the increment of the objective functional allows us to state the necessary optimality condition (analogously to results obtained in [3]).

Theorem. *If process $\{u, x\}$ is optimal in the problem under consideration, then*

$$\omega(t) = h_u(p(t), x(s_0, t), u(t), t) \cdot \dot{u}(t) = 0, \ \ t \in T,$$

where $p(t)$ is a solution to the adjoint problem (2.2).

4. A SMOOTH CONTROL IMPROVING METHOD

Let us describe the general scheme of the method based on the use of the stated optimality condition.

1) Choose an initial smooth control $u^0(t)$ satisfying the pointwise constraint (1.3). Assume that on the iteration k the control $u^k(t)$ is calculated.

2) For the control $u^k(t)$ find a solution $x^k(s, t)$ to the direct problem and a solution $\psi^k(s, t)$, $p^k(t)$ to the adjoint problem.

3) Using obtained solutions, calculate the value of the functional $J^k = J(u^k)$ and construct the function

$$\omega_k(t) = h_u(p^k(t), x^k(s_0, t), u^k(t), t) \cdot \dot{u}^k(t).$$

Verify the optimality condition $\omega_k(t) = 0$. If it is fulfilled, then the method stops.

4) If the calculated control violates the optimality condition, construct its smooth variation

$$u^k_{\varepsilon_k}(t) = u^k(t + \varepsilon_k\delta_k(t)), \ \ \delta_k(t) = \frac{(t - t_0)(t_1 - t)\omega_k(t)}{(t_1 - t_0) \max\limits_{t \in T} |\omega_k(t)|}.$$

The value of the parameter ε_k is determined as a result of the approximate solution to the following one-dimensional minimization problem:

$$\varepsilon_k : J(u^k_\varepsilon) \to \min, \ \ \varepsilon \in [0, 1].$$

If the calculated value of this parameter is close to zero, then there is no improvement of the functional on the method step.

5) Choose $u^{k+1}(t) = u^k_{\varepsilon_k}(t)$ as the next approximation and proceed with the iterative process. The stop criterion consists in the fulfillment (on some kth iteration of the method) of one of the following conditions.

a) The function $u^k(t)$ satisfies (with a given accuracy) the necessary optimality condition. For example, the value of the function $\omega_k(t)$ at each point $t \in T$ is close to zero if $\max\limits_{t \in T} |\omega_k(t)| \leq 10^{-5}$.

b) The value of the functional calculated on the previous iteration (that with the number $k-1$) is not improved, for example, $J^k - J^{k-1} > 10^{-6}$. The sequence of controls generated by the method is relaxation [3].

5. NUMERICAL TESTS

Consider the application of the described method to one test example. In the square $[0,1]\times[0,1]$ we consider the optimal control problem

$$x_t - x_{ss} = e^s \cos t,$$

$$x(s,0) = s + 0.3,\ \ x_s(1,t) = 0,\ \ x_t(0,t) = x(0,t)u(t),\ \ u(t) \in [0,3].$$

The objective functional takes the form

$$J(u) = \frac{1}{2}\int_S (x(s,t_1) - x^*(s))^2\, ds,$$

where the function $x^*(s) = x^*(s,t_1)$ is evaluated for the control $u^*(t) = 2 + \sin 5t$. The initial control is

$$u^0(t) = 1 + \cos 30t + \sin(13/4)t.$$

The value of the functional is $J(u^0) = 0.5849$.

We have solved the problem by the method described above and obtained the following results: the value of the objective functional on the procedure output is $J(u^k) = 0.0001381$, the optimality error is $\max\limits_{t\in T} |\omega_k(t)| = 0.030583$, the total number of iterations equals 32, the stop criterion consisted in attaining the given accuracy with respect to the functional value.

According to results of the numerical test, one can efficiently solve the optimal control problem with initial-boundary conditions for parabolic equations by the proposed technique (the smooth control improving method).

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REFERENCES

1. Demidenko, N. D. *Controlled Distributed Systems* (Nauka, Novosibirsk, 1999) [in Russian].
2. Vasil'ev, F. P. *Optimization Methods* (Faktorial Press, Moscow, 2002) [in Russian].
3. Arguchintsev, A. V. *Optimal Control over Hyperbolic Systems* (Fizmatlit, Moscow, 2007) [in Russian].

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